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THE HOMOTOPY TYPE OF THE SPACE OF DIFFEOMORPHISMS. I

BY

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ABSTRACT. A new proof is given of the unpublished results of Morlet on the relation between the homeomorphism group and the diffeomorphism group of a smooth manifold. In particular, the result $\mathrm{Diff}(D^n, \partial) \simeq \Omega^{n+1}(\mathrm{Top}_n/O_n)$ is obtained. The main technique is fibrewise smoothing.

0. Introduction. Throughout this paper "differentiable" means C^∞ differentiable. By a diffeomorphism, homeomorphism, pl homeomorphism, pd homeomorphism $f: M \to M$, identity on K, we will mean a diffeomorphism, homeomorphism, etc., which is homotopic to the identity among continuous maps $h: (M, \partial M) \to (M, \partial M)$, identity in K.

Given M a differentiable compact manifold, one considers $\mathrm{Diff}(M^n, K)$ and $\mathrm{Homeo}(M^n, K)$ the topological groups of all diffeomorphisms (homeomorphisms) which are the identity on K endowed with the C^∞- (C^0)-topology.

If K is a compact differentiable submanifold with or without boundary (possibly empty), $\mathrm{Diff}(M^n, K)$ is a C^∞-differentiable Fréchet manifold, hence

(a) $\mathrm{Diff}(M^n, K)$ has the homotopy type of a countable CW-complex,

(b) as a topological space $\mathrm{Diff}(M^n, K)$ is completely determined (up to homeomorphism) by its homotopy type (see [4] and [13]).

In order to compare $\mathrm{Diff}(M^n, K)$ with $\mathrm{Homeo}(M^n, K)$ (as also with the com binatorial analogue of $\mathrm{Homeo}(M^n, K)$), it is very convenient to consider the semisimplicial version of $\mathrm{Diff}(\cdots)$ and $\mathrm{Homeo}(\cdots)$, namely, the semisimplicial groups $S^d(\mathrm{Diff}(M^n, K))$ the singular differentiable complex of $\mathrm{Diff}(M^n, K)$ and $S\,\mathrm{Homeo}(M^n, K)$ the singular complex of $\mathrm{Homeo}(M^n, K)$. If $t: |M^n| \to M$ is a differentiable triangulation (see [26] or [21]) we can also consider the semisimplicial complexes $\mathrm{PL}(|M^n|, t^{-1}(K))$ and $\mathrm{PD}(|M^n|, t, K)$ where $\mathrm{PL}(|M^n|, t^{-1}(K))$ is

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a semisimplicial group. The k-simplexes are pl (piecewise linear) homeomorphisms

$$\begin{array}{ccc} \sigma\colon \Delta[k]\times|M^n| & \to & \Delta[k]\times|M^n| \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

respectively, pd (piecewise differentiable) homeomorphisms

$$\begin{array}{ccc} \sigma\colon \Delta[k]\times|M^n| & \to & \Delta[k]\times M^n \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

which commute with the first factor projection and restrict to the identity on $\Delta[k]\times t^{-1}(K)$. The right composite by t induces a semisimplicial injective map of the first semisimplicial complex into the second which is a homotopy equivalence.

The semisimplicial groups $S^d\mathrm{Diff}(\cdots)$, S Homeo$(\cdots)$, PL$(\cdots)$ and the semisimplicial complex PD$(\cdots)$ suggest one should consider simultaneously the semisimplicial[3] groups $\widetilde{\mathrm{Diff}}(M^n, K)$, $\widetilde{\mathrm{Homeo}}(M^n, K)$, $\widetilde{\mathrm{PL}}(|M^n|, t^{-1}K)$ and the simplicial complex $\widetilde{\mathrm{PD}}(M^n, t, K)$ whose k simplices are diffeomorphisms $h\colon \Delta[k]\times M\to\Delta[k]\times M$, homeomorphisms $h\colon \Delta[k]\times M\to\Delta[k]\times M$, pl homeomorphisms $h\colon\Delta[k]\times|M|\to\Delta[k]\times|M|$ or pd homeomorphisms $h\colon\Delta[k]\times|M|\to\Delta[k]\times M$ so that the following properties are satisfied:

(1) $h\,|\,\Delta[k]\times K=\mathrm{id}$, $h|\Delta[k]\times K=\mathrm{id}$ or $h|\Delta[k]\times t^{-1}(K)=t$,

(2) $h(d_i\Delta[k]\times M)\subset d_i\Delta[k]\times M$, $h(d_i\Delta[k]\times|M|)\subset d_i\Delta[k]\times M$ or $h(d_i\Delta[k]\times|M|)\subset d_i\Delta[k]\times M$.

The group structure of $\widetilde{\mathrm{Diff}}(\cdots)$, $\widetilde{\mathrm{Homeo}}(\cdots)$, $\widetilde{\mathrm{PL}}(\cdots)$ is given by composition. Surgery methods plus smoothing theory permit (at least from a theoretical point of view) the description of the homotopy type of $\widetilde{\mathrm{Diff}}$, $\widetilde{\mathrm{PL}}$, $\widetilde{\mathrm{Homeo}}$ (for homotopy groups see [2]). *Our principal aim here is to study* Diff(M^n, K) *and to get information about its homotopy type*. We do this by studying the relationship between different groups, Diff, Homeo, PL, $\widetilde{\mathrm{Diff}}$, $\widetilde{\mathrm{Homeo}}$, and $\widetilde{\mathrm{PL}}$. We use the results obtained by comparing these semisimplicial and simplicial groups to get information about Diff(M^n, K) and also about Top_n, PL_n, Top and PL. The present paper follows essentially these ideas. The main results are contained in §4 of Part I and §§5, 6, 7 of Part II, and they will be briefly reviewed below. In order to state the results it will be convenient to fix some notations. Let M^n be a differentiable manifold and $O(n)\to T^d\to M^n$ the principal tangent bundle

[3] These simplicial groups and complexes have no degeneracies.

of M. Because the inclusions $O_n \to \mathrm{Top}_n$, $O_n \to O$, $O_n \to \mathrm{Top}_n \to \mathrm{Top}$, are group homomorphisms, we can consider the principal bundle $\mathrm{Top}_n \to T^t \to M^n$ and also the principal bundles $O \to \overline{T}^d_s \to M^n$, $\mathrm{Top} \to \overline{T}^t_s \to M^n$ associated with $O_n \to T^d \to M^n$. We also consider the bundles $\mathrm{Top}_n/O_n \to P^t \to M^n$ and $\mathrm{Top}/O \to \overline{P}^t \to M^n$ associated in an obvious way. Because the bundles $T^t \to M$ and $\overline{T}^t \to M$, come from an O_n-principal bundle, the bundles $P^t \to M$ and $\overline{P}^t \to M$ have a natural continuous cross-section (up to homotopy) s. If M^n has nonempty boundary, the bundle $\mathrm{Top}_n/O_n \to P^t/\partial M \to \partial M$ contains the subbundle $\mathrm{Top}_{n-1}/O_{n-1} \to {}^{\partial}P^t \to \partial M$. We denote by $\Gamma(P^t, s)$, respectively $\Gamma(\overline{P}^t, s)$, the basepointed space of all continuous cross-sections of $P^t \to M$, respectively $\overline{P}^t \to M$, endowed with the C^0-topology and with s as base point.

We denote by $\Gamma(P^t, {}^{\partial}P^t, s)$ the subspace of $\Gamma(P^t, s)$ consisting of those cross-sections whose restrictions to ∂M are cross-sections of ${}^{\partial}P^t \to \partial M$; by $\Gamma^K(P^t, s)$, respectively $\Gamma^K(\overline{P}^t, s)$, the subspaces of $\Gamma(\cdots)$ consisting of those cross-sections which agree with s on K; and by $\Gamma^K(P^t, {}^{\partial}P^t, s)$ the intersection $\Gamma(P^t, {}^{\partial}P^t, s) \cap \Gamma^K(P^t, s)$.

Essentially the same constructions can be made in the PL case, modulo some technical difficulties due to the fact that there is no naturally defined map from O_n into $\{\mathrm{PL}_n\}$.(4) We use instead the natural inclusion of S^dO_n in PD_n. Under composition, S^dO_n acts freely on the left of PD_n, while PL_n acts freely on the right of PD_n (of course the sides in which these groups act can be reversed, by the usual trick). Writing O_n for S^dO_n, we construct the bundle P^{pl} as follows: Let S^{pl} be the principal semisimplicial tangent bundle of $|M|$. Since $\mathrm{PL}_n \to \mathrm{PD}_n$ is a homotopy equivalence, $S^{pl} \times_{\mathrm{PL}_n} \mathrm{PD}_n$ is homotopy equivalent to S^{pl}. Since the action of O_n on PD_n commutes with the action of PL_n, we may divide by O_n; and we set $P^{pl} = \{(S^{pl} \times_{\mathrm{PL}_n} \mathrm{PD}_n)/O_n\}$. Then we have the fibrations $\{\mathrm{PD}_n/O_n\} \to P^{pl} \to |M|$, and $\{\mathrm{PD}_n\} \to T^{pl} \to |M|$, where $T^{pl} = \{S^{pl} \times_{\mathrm{PL}_n} \mathrm{PD}_n\}$. The bundles $\overline{T}^{pl}_s$ and $\overline{P}^{pl}$ are constructed similarly. The spaces of cross-sections $\Gamma(P^{pl}, s)$ etc., are then defined.

The main result of §4 claims that for any $K^{n-1} \subset \partial M$, a compact differentiable submanifold of the boundary, there exist basepoint preserving continuous maps

(1) $i^t: \{S\,\mathrm{Homeo}(M^n, K)/S^d\mathrm{Diff}(M^n, K)\} \to \Gamma^K(P^t, {}^{\partial}P^t, s)$ and

(2) $i^{pl}: \{\mathrm{PD}(M^n, K)/S^d\mathrm{Diff}(M^n, K)\} \to \Gamma^K(P^{pl}, {}^{\partial}P^{pl}, s)$

which induce an injective correspondence for connected components and homotopy equivalence on any connected component (in case (1) $n \neq 4$).

This theorem (for $K = \partial M$ as also for $\partial M = \emptyset$) has been announced by C. Morlet in 1969 [20] (essentially it goes back to Cerf [6] who has conjectured and

(4) If X is a semisimplicial complex, we denote by $\{X\}$ its geometric realization.

proven, for $n = 3$, that $\pi_i(\mathrm{Diff}(D^n, \partial D^n)) = \pi_{i+n+1}(\mathrm{Top}_n/O_n))$, but since then no complete proofs have appeared. Partly, this work is our attempt to understand and prove Morlet's statement. Actually, it seems that the present proof is different from Morlet's ideas.

One should also notice that there exist basepoint preserving maps

(1′) $\tilde{i}^t\colon \{\widetilde{\mathrm{Homeo}}(M^n, \partial M^n)/\widetilde{\mathrm{Diff}}(M^n, \partial M^n)\} \to \Gamma^{\partial M}(\bar{P}^t, s)$ and

(2′) $\tilde{i}^{pl}\colon \{\widetilde{\mathrm{PD}}(M^n, \partial M^n)/\widetilde{\mathrm{Diff}}(M^n, \partial M^n)\} \to \Gamma^{\partial M}(\bar{P}^{pl}, s)$

with a similar property. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} i^t\colon \{S\,\mathrm{Homeo}(M^n, \partial M^n)/S^d\,\mathrm{Diff}(M^n, \partial M^n)\} & \longrightarrow & \Gamma^{\partial M}(P^t, s) \\ \downarrow & & \downarrow \\ \tilde{i}^t\colon \widetilde{\mathrm{Homeo}}(M^n, \partial M^n)/\widetilde{\mathrm{Diff}}(M^n, \partial M^n) & \longrightarrow & \Gamma^{M}(\bar{P}^t, s) \end{array}$$

and the corresponding commutative diagram involving i^{pl} and $\tilde{i}^{pl}$.

In Part II, we will give applications of Morlet's result. The main theorem of §4 enables us to extend results of Cerf and Wagoner-Hatcher to the pl and topological cases. More precisely, if M^n is a closed[5] differentiable manifold, $t\colon |M^n| \to M^n$ a differentiable triangulation and

$C^d(M^n) = \mathrm{Diff}(M^n \times [0, 1], M^n \times \{0\})$, $\quad C^t(M^n) = \mathrm{Homeo}(M^n \times [0, 1], M^n \times \{0\})$

and

$$C^{pl}(M^n) = \mathrm{PL}(|M^n| \times [0, 1], |M^n| \times \{0\})$$

then we derive $\pi_0(C^d(M^n)) \approx \pi_0(C^t(M^n)) \simeq \pi_0(C^{pl}(M^n))$ (the first isomorphism requires $n \neq 3, 4$.

To illustrate the usefulness of this result we recall, firstly, that Cerf (in the simply connected case) and later on Hatcher and Wagoner (in general) have computed $\pi_0(C^d(M^n))$, and, secondly, notice the existence of the following commutative diagram with exact rows.

$$\begin{array}{cccccc} \to \pi_1(\mathrm{Diff}(M^n)) & \to \pi_1(\widetilde{\mathrm{Diff}}(M^n)) & \to \bar{\pi}_0(C^d(M^n))^7 & \to \pi_0(\mathrm{Diff}(M^n)) & \to \pi_0(\widetilde{\mathrm{Diff}}(M^n)) & \to 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \to \pi_1(\mathrm{PL}(|M^n|)) & \to \pi_1(\widetilde{\mathrm{PL}}(|M^n|)) & \to \bar{\pi}_0(C^{pl}(M^n)) & \to \pi_0(\mathrm{PL}(|M^n|)) & \to \pi_0(\widetilde{\mathrm{PL}}(|M^n|)) & \to 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \to \pi_1(\mathrm{Homeo}(M^n)) & \to \pi_1(\widetilde{\mathrm{Homeo}}(M^n)) & \to \bar{\pi}_0(C^t(M^n)) & \to \pi_0(\mathrm{Homeo}(M^n)) & \to \pi_0(\widetilde{\mathrm{Homeo}}(M^n)) & \to 0 \end{array}$$

Another immediate corollary of the main result is $\mathrm{Diff}(D^n, \partial D^n) \sim \Omega^{n+1}(\mathrm{PL}_n/O_n)$[6] ($\sim$ means homotopy equivalent). This fact enables us to define

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[5] By closed manifold, we always mean a compact manifold without boundary.

[6] By abuse of notation we will sometimes write PL_n/O_n for $\{\mathrm{PD}_n/O_n\}$.

[7] $\bar{\pi}_0(C(M^n))$ is a quotient map of $\pi_0(C(M^n))$.

in §7 a new multilinear pairing

$$\pi_i(\mathrm{Diff}(D^n, \partial D^n)) \otimes \pi^S_{r_1} \otimes \cdots \otimes \pi^S_{r_k} \to \pi_{i+r_1+\cdots+r_s}(\mathrm{Diff}(D^n, \partial D^n))$$

for $r_1 < i + n, r_2 < i + n + r_1, \cdots, r_k < i + n + r_1 + \cdots + r_{k-1}$, where π^S_j denotes the jth stable homotopy group of the sphere S^0. This pairing allows us to check for many pairs of indices (i, n) that $\pi_i(\mathrm{Diff}(D^n, \partial D^n))$ contains $\mathbf{Z}_p$. These computations improve considerably the list of nontrivial homotopy $\pi_i(\mathrm{Diff}(D^n, \partial D^n))$ (see [3]), and also give the first examples of nontrivial elements in $\pi_i(\mathrm{Diff}(D^n, \partial D^n))$ which survive in Γ_{n+i+1} for $i > n$ (as also for $i > kn$ for any integer k).

In §5 we discuss the homotopy properties of Top_n, PL_n, Top, and PL and give some new results and conjectures about their homotopy groups.

The major new technique of our proof is fibrewise smoothing in §2. It enables us to prove in particular that a fibrewise smoothing of a topological bundle is a differentiable bundle (Theorem 2.6).

1. The semisimplicial complexes of embeddings (E^-), immersion (Im^-), and bundle representations (R^-). Let M^n be a compact connected differentiable manifold with nonempty boundary, P^n an interior collar of ∂M^n (i.e. P is the image of a differentiable imbedding of $\partial M \times [0, 1]$ in M which restrict to the identity on ∂M). We denote by $\widetilde{M}$ the open differentiable manifold $M \cup \partial M \times [0, \infty)$ and by $M_\alpha = M \cup \partial M \times [0, \alpha]$. Let $t\colon |M^n| \to M$ be a differentiable triangulation ($|M^n|$ being a pl-manifold) so that $|P^n| = t^{-1}(P)$ is a pl interior collar. We define

$$|\widetilde{M}| = |M| \cup \partial|M| \times [0, \infty) \quad \text{and} \quad |M_\alpha| = |M| \cup \partial|M| \times [0, \alpha]$$

and extend t to a differentiable triangulation $\tilde{t}\colon |\widetilde{M}| \to \widetilde{M}$.

We introduce the *semisimplicial groups* $\mathrm{Diff}(M^n, P)$, $\mathrm{Homeo}(M^n, P)$, $\mathrm{PL}(|M^n|, |P|)$ and the *semisimplicial complex* $\mathrm{PD}(M^n, t; P)$.

The k-simplexes of these complexes are diffeomorphisms, homeomorphisms, pl homeomorphisms or pd homeomorphisms

$$\begin{array}{ccc} \sigma\colon \Delta[k] \times M & \longrightarrow & \Delta[k] \times M \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array} \qquad \begin{array}{ccc} \sigma\colon \Delta[k] \times |M| & \longrightarrow & \Delta[k] \times |M| \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

$$\begin{array}{ccc} \sigma\colon \Delta[k] \times |M| & \longrightarrow & \Delta[k] \times M \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

which agree with the identity (respectively with t in the pd case) on a neighborhood of $\Delta[k] \times P$. The operators s_i and d_i are obviously defined, and the composition makes Diff( ), Homeo( ), PL( ), semisimplicial groups.

Let N^n be an open differentiable manifold and $t'\colon |N| \to N$ a differentiable triangulation; we define the semisimplicial complexes

$$\mathrm{E}^d(M^n, N), \quad \mathrm{E}^t(M^n, N), \quad \mathrm{E}^{pl}(|M^n|, |N^n|), \quad \mathrm{E}^{pd}(|M^n|, N, t')$$

and

$$\mathrm{Im}^d(M^n, N), \quad \mathrm{Im}^t(M^n, N), \quad \mathrm{Im}^{pl}(|M^n|, |N|), \quad \mathrm{Im}^{pd}(|M^n|, N)$$

as follows:

A k-simplex of $\mathrm{E}^d(M^n, N)$ or $\mathrm{Im}^d(M^n, N)$ is an equivalence class of differentiable embeddings or immersions

$$\begin{array}{ccc} \sigma\colon \Delta[k] \times M_\alpha & \longrightarrow & \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

two such (σ_1, α_1), (σ_2, α_2) are equivalent if and only if σ_1 and σ_2 agree on $\Delta[k] \times M_\alpha$ with $\alpha < \alpha_1, \alpha_2$. Analogously we define $\mathrm{E}(\cdots)$ and $\mathrm{Im}(\cdots)$ for the exponents t, pl and pd (s_i's and d_i's are obviously defined). We will also need the semisimplicial complexes

$$\mathrm{R}^d(M^n, N), \quad \mathrm{R}^t(M^n, N), \quad \mathrm{R}^{pl}(|M^n|, |N^n|), \quad \mathrm{R}^{pd}(|M^n|, t; N^n).$$

$\mathrm{R}^d(M^n, N)$. A k-simplex is a vector bundle representation $\tilde{f}$, f

$$\begin{array}{ccc} \Delta[k] \times T(M) & \xrightarrow{\tilde{f}} & \Delta[k] \times T(N) \\ \downarrow & & \downarrow \\ \Delta[k] \times M & \xrightarrow{f} & \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

i.e. $\tilde{f}$ is a linear isomorphism in any fibre; ($T(M)$ and $T(N)$ denote here the tangent vector bundle of M and N).

$\mathrm{R}^t(M^n, N)$. A k-simplex is a germ of topological microbundle representation

$$\begin{array}{ccc} b\colon \Delta[k] \times T(M^n) & \longrightarrow & \Delta[k] \times T(N^n) \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

(see also [17], [18]).

$\mathrm{R}^{pl}(|M^n|, |N|)$ *and* $\mathrm{R}^{pd}(|M^n|, N)$. A k-simplex is a germ of pl (pd) microbundle representations

$$\begin{array}{ccc} h: \Delta[k]\times T(|M^n|) & \to & \Delta[k]\times T(|N^n|) \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array} \qquad \begin{array}{ccc} h: \Delta[k]\times T(|M^n|) & \to & \Delta[k]\times TN \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

with $T(|M^n|)$ and $T(|N^n|)$ the pl microbundles of $|M^n|$ and $|N|$ (for microbundle terminology we refer to [17]); for more details see ([9], [17]).

The restriction to P (respecitvely to $|P|$) defines the semisimplicial maps r^d, r^t, r^{pl}, r^{pd} in the following commutative diagram (Figure 1). The unmarked arrows are inclusions. The t_* are injective semisimplicial maps induced by right composition with t and the t'^* are injective semisimplicial maps induced by left composition by t'.

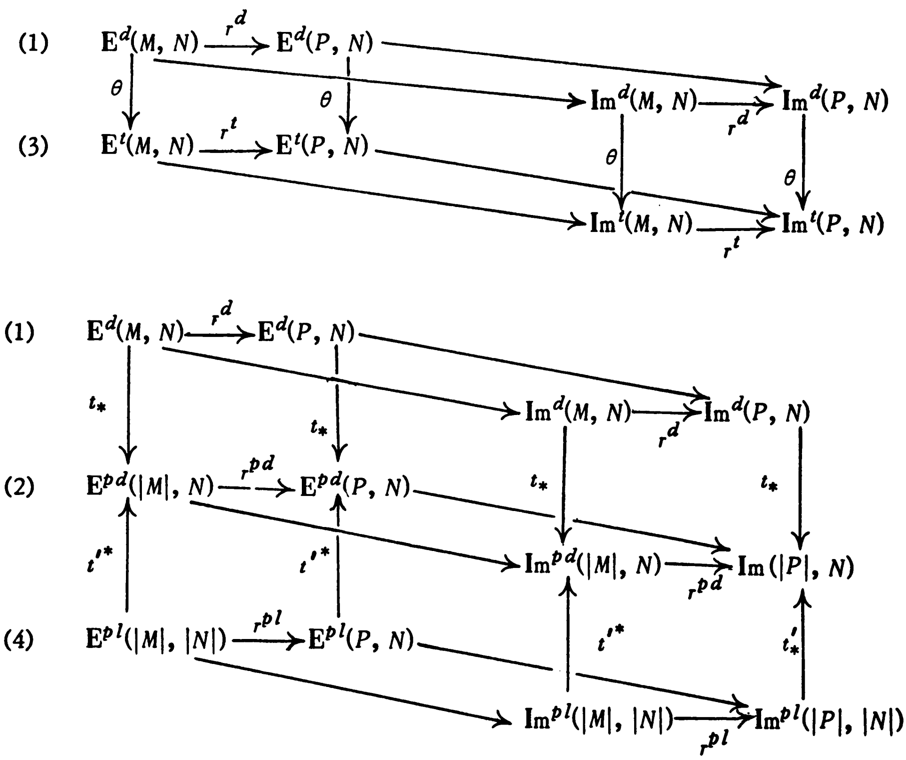


Figure 1

We also have the following commutative diagram (Figure 2) when δ^d, δ^t are the "differential" and r^d, r^t, θ are as before.

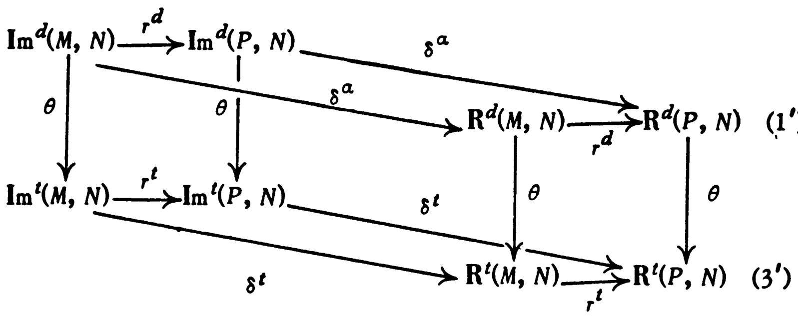


Figure 2

Similarly, we have the corresponding pl-pd diagram, involving δ^{pd}, δ^{pl} along with δ^d, which we will not display, but which we denote by (2′) and (4′).

Theorem 1.1. (a) *All semisimplicial complexes* $\mathrm{E}^{\cdots}$, $\mathrm{Im}^{\cdots}$, $\mathrm{R}^{\cdots}$ *are Kan complexes.*

(b) *The lines* (1), (3), (4) *of Figure* 1 *are principal Kan fibrations of semisimplicial groups* $\mathrm{Diff}(M^n, P^n)$, $\mathrm{Homeo}(M^n, P^n)$, $\mathrm{PL}(|M^n|, |P^n|)$ *and the lines* (2), (1′), (2′), (3), (4′) *are Kan fibrations.*

(c) *The semisimplicial maps* t'^* *are all homotopy equivalences.*

(d) *The semisimplicial maps* $\delta^d, \delta^{pd}, \delta^{pl}, \delta^t$ *are all homotopy equivalences.*

Theorem 1.1 is a collection of well-known facts. (Partly these statements can be found in the literature in a slightly different context.)

(a) All semisimplicial complexes $\mathrm{E}^{\cdots}$, $\mathrm{Im}^{\cdots}$, $\mathrm{R}^{\cdots}$ with indices *pl*, *pd*, and *t* are Kan because one can always find a pl retraction of $\Delta[k]$ on $\Lambda^i[k]$ where $\Lambda^i[k] = \dot{\Delta}[k] \setminus \mathrm{Int}(d_i \Delta[k])$.

$\mathrm{R}^d(\ )$ and $\mathrm{Im}^d(\ )$ are Kan by Lemmas 1.2 and 1.3 below.

(b) (1), (3), and (4) are principal Kan fibrations because $\mathrm{Diff}(M^n, P^n)$, $\mathrm{PL}(|M^n|, |P^n|)$ and $\mathrm{Homeo}(M^n, P^n)$ act freely on $\mathrm{E}^d(M^n, N^n)$, $\mathrm{E}^{pl}(|M^n|, |N^n|)$ and $\mathrm{E}^t(M^n, N^n)$ and by the ambient isotopy theorem in the differential pl or topological case (for the statement of the ambient isotopy theorem, see §2 below), the cosets of these actions identify to a union of connected components of $\mathrm{E}^d(P^n, N^n)$, $\mathrm{E}^{pl}(|P^n|, |N^n|)$ and $\mathrm{E}^t(P^n, N^n)$. (2) is a Kan fibration as a consequence of the pd ambient isotopy theorem which can be easily observed from the pl ambient isotopy theorem and differential triangulation [14], [21], or [26].

(1′) is a Kan fibration by the extension property of continuous sections and by Lemmas 1.2 and 1.3. For (3′) and (4′) see [18], respectively [8], and (2′) follows from (3′).

(c) is a consequence of triangulation techniques of J. H. Whitehead (see [26], [21], or [3]).

(d) δ^d, δ^{pl} are homotopy equivalences by immersion theory in the differentiable case due to Smale and Hirsch (see [10]), in the pl case to Haefliger-Poenaru [9], and in the topological case to J. Lees [18] or [19]. Actually the semisimplicial complexes Im^d and R^d are not the singular complexes of the spaces of immersions and bundle maps, but they have the same homotopy type because of Lemmas 1.2 and 1.3.

If P is the image of a differentiable embedding $b: \partial M \times [0, 1] \to M$ we consider $\widetilde{b}: \partial M \times [0, \infty) \to M$ a differentiable embedding which extends b and denote by P_α the image of $\partial M \times [0, \alpha]$, $\alpha > 1$, by $\widetilde{b}$.

Lemma 1.2. (a) $\mathrm{Diff}(M^n, P^n) = \mathrm{inj}\lim_{\alpha \to 1} S^d \mathrm{Diff}(M^n, P^n_\alpha)$ *with* $S^d(\mathrm{Diff}(M^n, P_{\alpha'})) \to S^d(\mathrm{Diff}(M^n, P_{\alpha''}))$ *the obvious semisimplicial inclusion induced by the continuous inclusion* $\mathrm{Diff}(M^n, P_{\alpha'}) \subset \mathrm{Diff}(M^n, P_{\alpha''})$, $\alpha'' < \alpha'$.

(b) *If* $\mathrm{Im}(M, N)$, $E(M, N)$ *denote the differentiable Fréchet manifolds of* C^∞-*differentiable immersions, respectively embeddings endowed with the* C^∞-*topology, then* $\mathrm{Im}^d(M, N) = \mathrm{inj}\lim_{\alpha \to 0} S^d \mathrm{Im}(M_\alpha, N)$ *and* $\mathrm{E}^d(M, N) = \mathrm{inj}\lim_{\alpha \to 0} S^d E^d(M, N)$.

Lemma 1.3. (a) *If* X_i *is a sequence of Kan complexes and* $p_i: X_i \to X_{i+1}$ *a sequence of semisimplicial maps then* $X = \mathrm{inj}\lim_i X_i$ *is a Kan complex.*

(b) *If all* p_i *are homotopy equivalences, the semisimplicial maps* $p^i: X_i \to X$, $p^i = \lim_{k \to \infty} p_{i+k} \cdots p_{i-1} p_i$ *are homotopy equivalences.*

(c) *Let* $f_i: X_i \to Y_i$ *be a sequence of Kan fibrations,* $p_i^X: X_i \to X_{i+1}$ *and* $p_i^Y: Y_i \to Y_{i+1}$ *semisimplicial maps so that the diagrams*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ {\scriptstyle p_i^X}\downarrow & & \downarrow{\scriptstyle p_i^Y} \\ X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} \end{array}$$

are commutative. Then $f: X = \mathrm{inj}\lim X_i \to \mathrm{inj}\lim Y_i$ *is a Kan fibration.*

If $N = \widetilde{M}$, the inclusions $P \subset \widetilde{M}$ and $|P| \subset |\widetilde{M}|$ define a basepoint (0-simplex) in $\mathrm{E}^d(P, \widetilde{M})$, $\mathrm{E}^t(P, \widetilde{M})$, $\mathrm{E}^{pl}(|P|, |\widetilde{M}|)$ and their differential a basepoint in $\mathrm{R}^d(P, \widetilde{M})$, $\mathrm{R}^t(P, \widetilde{M})$ and $\mathrm{R}^{pl}(|P|, |\widetilde{M}|)$. Similarly, the triangulation t restricted to $|P|$ defines a basepoint in $\mathrm{E}^{pd}(|P|, \widetilde{M})$ and $\mathrm{R}^{pd}(|P|, \widetilde{M})$. All these basepoints will be denoted by $*$.

Combining the diagrams (Figure 1 and Figure 2) and taking the fibres corresponding to $*$ we get the commutative diagram (Figure 3)

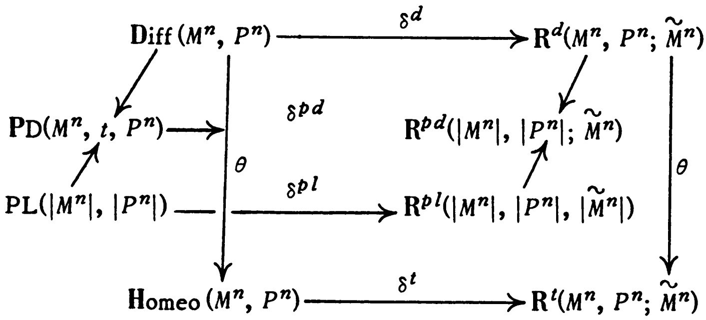


Figure 3

where $R^{\cdots}(M^n, P^n; \tilde{M}^n)$ denote the fibres of (1′), (2′), (3′) and (4′) corresponding to the basepoint *.

We note also that $\delta^{\cdots}$ in the diagram (Figure 3) factors through $i^{\cdots}$: $R^{\cdots}(M^n, P, M^n) \to R^{\cdots}(M^n, P, \tilde{M}^n)$, the semisimplicial maps induced by the inclusion $M^n \subset \tilde{M}^n$ and $|M^n| \subset |\tilde{M}^n|$ and $i^{\cdots}$ are homotopy equivalences. Therefore we have the following commutative diagram

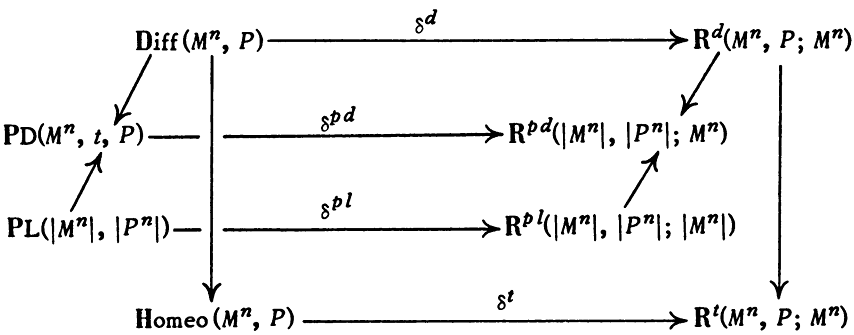


Figure 3′

where $R^{\cdots}(M^n, P; M^n)$ are the semisimplicial subcomplexes whose simplexes are germs of bundle representations which agree with the identity on P. It is clear that $R^d(M^n, P; M^n)$, $R^t(\cdots)$ and $R^{pl}(\cdots)$ are semisimplicial semigroups and δ^d, δ^t, and δ^{pl} are homomorphisms. The semisimplicial group Diff(M^n, P) acts freely (on the left) of PD(M^n, t, P) and the semisimplicial semigroup $R^d(M^n, P; M^n)$ acts on $R^{pd}(|M^n|, |P^n|; M^n)$ and δ^d, δ^{pd} are compatible with these actions.

Remark. The diagram (Figure 3′) can be defined for any compact subset P of M^n, in particular for any compact differentiable submanifold of M^n possibly with boundary.

If $\partial M^n \neq \emptyset$, $\tau(M^n)/\partial M^n$ identifies to $\tau(\partial M^n) \oplus \epsilon$, ϵ being the trivial line bundle. When $N \not\supset \partial M$ we also need to consider the subcomplexes ${}^{\partial}\mathbf{R}^d(M^n, N; M^n)$, ${}^{\partial}\mathbf{R}^t(M^n, N; M^n)$, ${}^{\partial}\mathbf{R}^{pl}(|M^n|, |N^n|; M^n)$ and ${}^{\partial}\mathbf{R}^{pd}(|M^n|, |N^n|; M^n)$, $(|N^n| = t^{-1}(N^n))$ of $\mathbf{R}^d(\cdots)$, $\mathbf{R}^t(\cdots)$, $\mathbf{R}^{pl}(\cdots)$ and $\mathbf{R}^{pd}(\cdots)$ consisting of those simplexes

$$\begin{array}{ccc} \sigma\colon \Delta[k] \times \tau(M) & \longrightarrow & \Delta[k] \times \tau(M) \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

of $\mathbf{R}^{\cdots}(M, N; M)$ which satisfy

(a) $\sigma(\Delta[k] \times \tau(M)|\Delta[k] \times \partial M) \subset \Delta[k] \times \tau(M)|\Delta[k] \times \partial M$ and

(b) $\sigma|_{\Delta[k]\times\partial M} = \sigma^1 \times \mathrm{id}$ with σ^1 a k-simplex of $\mathbf{R}^{\cdots}(\partial M; \partial M)$.

It is clear that if $\partial M \subset N$, ${}^{\partial}\mathbf{R}^{\cdots}(\cdots) \equiv \mathbf{R}^{\cdots}(\cdots)$. The reader could check without any trouble that these semisimplicial complexes are Kan *semisimplicial semigroups* (except in the pd case).

Now we define the semisimplicial complexes ${}^{\partial}\underline{\mathbf{R}}^{\cdots}(\cdots)$ and ${}^{\partial}\overline{\mathbf{R}}^{\cdots}(\cdots)$, ${}^{\partial}\underline{\mathbf{R}}^{\cdots}(\cdots) \subseteq {}^{\partial}\overline{\mathbf{R}}^{\cdots}(\cdots) \subseteq {}^{\partial}\mathbf{R}^{\cdots}(\cdots)$ as consisting of those simplexes of ${}^{\partial}\mathbf{R}^{\cdots}(\cdots)$ which restrict on $\Delta[k] \times M$ to the identity respectively, homeomorphisms. The subcomplexes ${}^{\partial}\underline{\mathbf{R}}^{\cdots}(\cdots)$, ${}^{\partial}\overline{\mathbf{R}}^{\cdots}(\cdots)$ except ${}^{\partial}\underline{\mathbf{R}}^{pd}(\cdots)$ and ${}^{\partial}\overline{\mathbf{R}}^{pd}(\cdots)$ are semisimplicial groups (subgroups of ${}^{\partial}\mathbf{R}^{\cdots}(\cdots)$), $\delta^{\cdots}$ factors through ${}^{\partial}\overline{\mathbf{R}}^{\cdots}(\cdots) \subset {}^{\partial}\mathbf{R}^{\cdots}(\cdots) \subseteq \mathbf{R}^{\cdots}(\cdots)$, $\delta^t, \delta^d, \delta^{pl}$ are group homomorphisms, ${}^{\partial}\overline{\mathbf{R}}^d(\cdots)$, ${}^{\partial}\underline{\mathbf{R}}^d(\cdots)$ act freely on ${}^{\partial}\overline{\mathbf{R}}^{pd}(\cdots)$, ${}^{\partial}\underline{\mathbf{R}}^{pd}(\cdots)$ and these actions together with the action of $\mathrm{Diff}(M^n, N)$ on $\mathrm{PD}(|M^n|, |N^n|; M^n)$ are compatible by (δ^d, δ^{pd}). We have the commutative diagram (Figure 3″)

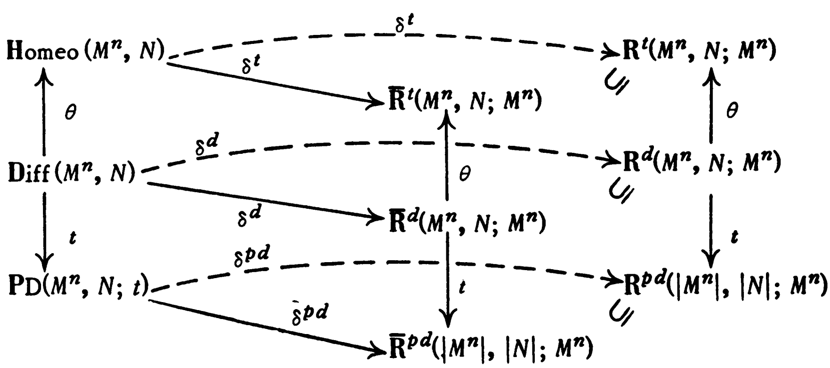


Figure 3″

Proposition 1.4.

(1) $\quad {}^{\partial}\underline{\mathbf{R}}^t(M^n, N; M^n)/{}^{\partial}\underline{\mathbf{R}}^d(M^n, N; M^n) \approx {}^{\partial}\overline{\mathbf{R}}^t(M^n, N; M^n)/{}^{\partial}\overline{\mathbf{R}}^d(M^n, N; M^n).$

(2) ${}^{\partial}\underline{\mathrm{R}}^{pd}(|M^n|, |N|; M^n)/{}^{\partial}\underline{\mathrm{R}}^{d}(M^n, N; M^n) \approx {}^{\partial}\mathrm{R}^{pd}(|M^n|, |N|; M^n)/{}^{\partial}\mathrm{R}^{d}(M^n, N; M^n)$.

(3) *For any basepoint* $x \in {}^{\partial}\mathrm{R}^d(M, N; M)$ *the inclusion of pairs*

$$({}^{\partial}\overline{\mathrm{R}}^t(M^n, N; M), \theta({}^{\partial}\overline{\mathrm{R}}^d(M^n, N; M))) \subseteq ({}^{\partial}\mathrm{R}^t(M^n, N; M^n), \theta({}^{\partial}\mathrm{R}^d(M^n, N; M^n)))$$

induces an isomorphism for homotopy groups (sets) in dimension $i \geq 1$.

(4) *For any basepoint* $x \in {}^{\partial}\overline{\mathrm{R}}^d(M^n, N; M^n)$ *the inclusion*

$$({}^{\partial}\overline{\mathrm{R}}^{pd}(|M^n|, |N|, M^n), t_* {}^{\partial}\overline{\mathrm{R}}^d(M^n, N^n, M^n))$$
$$\subseteq ({}^{\partial}\mathrm{R}^{pd}(|M^n|, |N|; M^n), t_* {}^{\partial}\mathrm{R}^d(M^n, N^n; N^n))$$

induces an isomorphism for homotopy groups (sets) in dimension ≥ 1.

Proof. We denote by Maps(M^n, N), respectively Maps(M^n, t, N), the space of all continuous maps from $(M, \partial M)$, respectively from $(|M^n|, \partial|M^n|)$, in $(M, \partial M)$ which are identity on N, respectively agree with t on N, endowed with the compact open topology. The right composite by t^{-1} defines a homeomorphism between Maps(M^n, N) and Maps(M^n, t, N). We consider the following diagram whose lines are Kan fibrations.

$$\begin{array}{ccc}
{}^{\partial}\mathrm{R}^t(M^n, N; M^n) & \xrightarrow{\quad p^t \quad} & S\,\mathrm{Maps}(M, N) \\
\uparrow \theta & & \parallel\!| \\
{}^{\partial}\mathrm{R}^d(M^n, N; M^n) & \xrightarrow{\quad p^d \quad} & S\,\mathrm{Maps}(M, N) \\
\downarrow t_* & & \uparrow t_*^{-1}(\text{isomorphism}) \\
{}^{\partial}\mathrm{R}^{pd}(|M^n|, |N|; M^n) & \xrightarrow{\quad p^{pd} \quad} & S\,\mathrm{Maps}(M, t; N)
\end{array}$$

It is clear that the lines of the diagram

$$\begin{array}{lccc}
(1) & {}^{\partial}\overline{\mathrm{R}}^t(M^n, N; M^n) & \xrightarrow{\quad p^t \quad} & \mathrm{Homeo}(M^n, N) \\
 & \theta\uparrow \qquad \mathrm{I} & & \parallel \\
(2) & {}^{\partial}\overline{\mathrm{R}}^d(M^n, N; M^n) & \xrightarrow{\quad p^d \quad} & \mathrm{Homeo}(M^n, N) \\
 & \downarrow \qquad \mathrm{II} & & \parallel \\
(3) & {}^{\partial}\overline{\mathrm{R}}^{pd}(M^n, N, N^n) & \xrightarrow{\quad t_*^{-1} \cdot p^{pd} \quad} & \mathrm{Homeo}(M^n, N)
\end{array}$$

are the pull-backs by Homeo$(M^n, N) \subseteq$ Maps(M^n, N) of the fibrations p^t, p^d, $t_*^{-1} \cdot p^{pd}$, as also that (1) and (2) are principal fibrations with fibres ${}^{\partial}\underline{\mathrm{R}}^{\cdots}(\cdots)$. This implies immediately (1) and (3) if we notice that in square I all the complexes are groups and all the maps group-homomorphisms, as also (4). If we carefully regard the square II and recall that ${}^{\partial}\underline{\mathrm{R}}^d(\cdots)$, ${}^{\partial}\overline{\mathrm{R}}^d(\cdots)$ and Diff(M^n, N) operate freely on ${}^{\partial}\underline{\mathrm{R}}^{pd}(\cdots)$, ${}^{\partial}\overline{\mathrm{R}}^{pd}(\cdots)$ and PD(M^n, N) and these operations are compatible with the maps p^d and $t_*^{-1} \cdot p^{pd}$, we check (2) also. Q.E.D.

We also have the following commutative diagram (Figure 4)

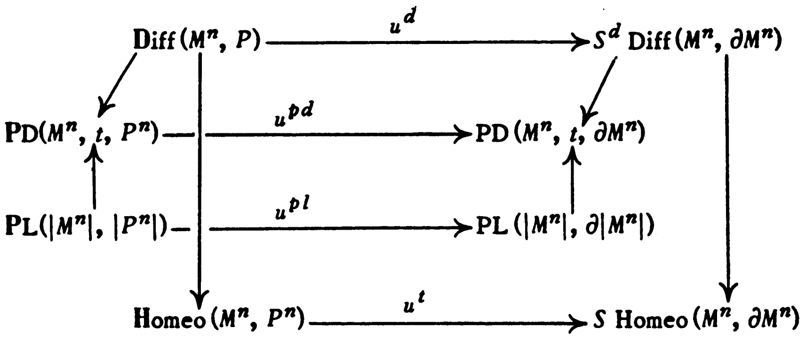


Figure 4

where $u^{\cdots}$ are the obvious semisimplicial inclusions.

Proposition 1.5. u^d, u^{pd}, u^{pl}, u^t *are homotopy equivalences.*

Proposition 1.5 follows from the unicity of tubular neighborhoods (here called neighborhoods) with parameters and Lemma 1.2 (see also [3]).

We also should notice that for N any compact differentiable submanifold of M^n we have

Proposition 1.6. *If* $k \geq n-1$ *then*

$$\mathrm{Diff}(M^n, N^k) \xrightarrow{u^d} S^d\mathrm{Diff}(M^n, N^k)$$

$$\mathrm{Homeo}(M^n, N^k) \xrightarrow{u^t} S\,\mathrm{Homeo}(M^n, N^k)$$

$$\mathrm{PD}(|M^n|, N^k) \xrightarrow{u^{pd}} \mathrm{PD}(M^n, N^k)$$

$$\mathrm{PL}(|M|^n, |N^k|) \xrightarrow{u^{pl}} \mathrm{PL}(|M|^n, |N^k|)$$

are homotopy equivalences.

2. **Fibrewise smoothings.** A topological, respectively pl smoothing on $I^k \times \widetilde{M}$ is a differentiable structure θ on the topological manifold $I^k \times \widetilde{M}$ (one forgets the differentiable structure of $I^k \times \widetilde{M}$), respectively a differentiable structure θ on the PL manifold $I^k \times |\widetilde{M}|$, so that $I^k \times |\widetilde{M}| \xrightarrow{\mathrm{id} \times |t|} (I^k \times \widetilde{M})_\theta$ is a differentiable triangulation.

Definition 2.1. *A smoothing* θ *on* $I^k \times M$ *is called a k-topological fibrewise smoothing, respectively a k-pl fibrewise smoothing, if*

(1) *θ is a topological, respectively pl, smoothing,*

(2) *$p:(I^k \times \widetilde{M})_\theta \to I^k$, the first factor projection, is a differentiable submersion.*

(3) *The differentiable structure induced on $\{0\} \times \widetilde{M}$ is concordant to the initial differentiable structure of $\widetilde{M}$.*

Remark. If K is a differentiable submanifold of I^k, $p^{-1}(K)$ is a differentiable submanifold of $(I^k \times \widetilde{M})_\theta$, respectively of $(I^k \times |\widetilde{M}|)_\theta$, which will be denoted by $(K \times \widetilde{M})_\theta$, respectively $(K \times |\widetilde{M}|)_\theta$.

It is easy to see (using smoothing theory for topological manifolds [18] and for pl manifolds [16]) that $(t \times \widetilde{M}^n)_\theta$ and $(0 \times \widetilde{M}^n)_\theta$, respectively $(t \times |\widetilde{M}^n|)_\theta$ and $(0 \times |\widetilde{M}^n|)_\theta$, viewed as differentiable structures on the same topological manifold $\widetilde{M}^n$, $n \neq 4$, respectively pl manifold $|\widetilde{M}|$, are concordant. Therefore, for any $\alpha_1 < \alpha^1 < \alpha_2 < \alpha^2 < \alpha_3$ and $t \in I^k$ one can find a differentiable embedding $\phi: M^n_{\alpha^2} \to (\{t\} \times \widetilde{M}^n)_\theta$ $(n \neq 4)$, respectively $\phi: M_{\alpha^2} \to (\{t\} \times |\widetilde{M}^n|)_\theta$, so that $\phi(M_{\alpha^i}) \subset \operatorname{Int} M_{\alpha_{i+1}}$ and $M_{\alpha^i} \subset \phi(M_{\alpha_i})$, respectively $\phi(M_{\alpha^i}) \subset \operatorname{Int} |M_{\alpha_{i+1}}|$ and $|M_{\alpha^i}| \subset \operatorname{Int} \phi(M_{\alpha^i})$. This means Theorem 2.1 below is true for $k = 0$ (i.e., $I^0 = \{0\}$).

Theorem 2.2. *If θ is a k-topological fibrewise smoothing, $n \neq 4, 5$ or $n = 5$ and $\partial M^5 = S^4$, respectively θ is a pl fibrewise smoothing and $0 < \alpha_1 < \alpha^1 < \alpha_3$, then there exists a differentiable embedding*

$$\begin{array}{ccc} \phi: I^k \times M_{\alpha^1} & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

so that

(1) $\phi(I^k \times M_{\alpha^1}) \subset \operatorname{Int} I^k \times M_{\alpha_3}$, $I^k \times M_{\alpha_1} \subset \operatorname{Int} \phi(I^k \times M_{\alpha^1})$,

(2) $\phi(\{0\} \times M_{\alpha^1}) \setminus \{0\} \times \operatorname{Int} M_{\alpha_1}$ *is homeomorphic to* $\partial M \times [0, 1]$ *respectively pd homeomorphic to* $\partial|M| \times [0, 1]$.

(3) *If*

$$\begin{array}{ccc} \psi: I^{k'} \times \{0\} \times M_{\alpha^1} & \to & (I^{k'} \times \{0\} \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times \{0\} & \end{array}$$

is a differentiable embedding compatible with (1) *and* (2), *then one can choose ϕ so that $\phi \mid I^{k'} \times \{0\} \times M_{\alpha^1} = \psi$.*

Remark to Theorem 2.2. One can replace $I^{k'} \times \{0\}$ in (3) by $I^{k'} \times \{0\} \cup I^{k'} \times \{1\} \cup \{0\} \times I^{k-k'}$.

We will prove Theorem 2.2 only in the topological case; as the proof is entirely formal the reader will have no difficulty in completing the proof in the pl case.

In order to prove Theorem 2.2 we need Propositions 2.3 and 2.4.

Proposition 2.3. *Let*

$$\begin{array}{ccc} \psi, \phi: I^k \times M_\alpha & \longrightarrow & (I^k \times \tilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

be differentiable embeddings so that

(1) $\phi(I^k \times M_\alpha) \subset \operatorname{Int} \psi(I^k \times M_\alpha)$,

(2) $\psi(\{x\} \times M_\alpha) \setminus \operatorname{Int} \phi(\{x\} \times M_\alpha)$ *is diffeomorphic to* $\partial M_\alpha \times [0, 1]$ *for some* $x \in I^k$.

Then there exist a differentiable embedding

$$\begin{array}{ccc} \tilde{\phi}: I^k \times M_{\alpha'} & \longrightarrow & (I^k \times \tilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

$\alpha' > \alpha$, *extending* ϕ *and a diffeomorphism*

$$\begin{array}{ccc} h: I^k \times M_{\alpha'} & \longrightarrow & I^k \times M_\alpha \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

so that $\tilde{\phi} \cdot h^{-1} = \psi$.

Proof. Consider

$$\begin{array}{ccc} \gamma = \psi^{-1} \cdot \phi: I^k \times M_\alpha & \longrightarrow & I^k \times M_\alpha \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

which by hypothesis satisfies $\gamma(I^k \times M_\alpha) \subset \operatorname{Int} I^k \times M_\alpha$. The differential ambient isotopy theorem gives a diffeomorphism

$$\begin{array}{ccc} \delta: I^k \times M_\alpha & \longrightarrow & I^k \times M_\alpha \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

so that $\delta \cdot (\mathrm{Id} \times \gamma_x) = \gamma$ and $\delta_x = \mathrm{id}$ where δ_x and γ_x denote the restrictions of δ and γ to $\{x\} \times M_\alpha$. We use again the hypothesis, namely $\{x\} \times M_{\alpha'} \setminus \gamma(\{x\} \times M_\alpha)$ is diffeomorphic to $\partial M_\alpha \times [0, 1]$, and extend γ_x to a diffeomorphism $\tilde{\gamma}_x: M_{\alpha'} \to M_\alpha$, so that $\tilde{\gamma}_x \mid M_\alpha = \gamma_x$. We take $\tilde{\phi} = \psi \cdot \delta \cdot (\mathrm{id} \times \tilde{\gamma}_x)$, and $h = \delta(\mathrm{id} \times \tilde{\gamma}_x)$. Q.E.D.

We denote by $I_0 = [1/4, 3/4]$, $I_1 = [0, 3/4]$ and $I_2 = [1/4, 1]$.

Proposition 2.4. *Let*

$$\begin{array}{ccc} \phi_i: I^k \times I_i \times M_\alpha & \longrightarrow & (I^k \times I_i \times \tilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times I_i & \end{array}$$

$i = 1, 2$, *be two differentiable embeddings such that the restrictions* $\phi_{(i)} = \phi_i \mid I^k \times I_0 \times M_\alpha$ *satisfy*

(1) $\phi_{(i)}(I^k \times I_0 \times \mathrm{Int}\, M_\alpha) \supset I^k \times I_0 \times M_{\alpha_1}$, $\alpha_1 < \alpha$,

(2) $\phi_{(i)}(I^k \times I_0 \times M_\alpha) \subset I^k \times I^0 \times \mathrm{Int}\, M_{\alpha_2}$, $\alpha_2 > \alpha$,

(3) $\phi_{(1)}(I^k \times I_0 \times M_\alpha) \subset \mathrm{Int}\, \phi_{(2)}(I^k \times I_0 \times M_\alpha)$,

(4) *there exists* $y \in I^k \times I_0$ *with* $\phi_2(\{y\} \times M_\alpha) \setminus \phi_1(\{y\} \times \mathrm{Int}\, M_\alpha)$ *diffeomorphic to* $\partial M_\alpha \times [0, 1]$.

Then, there exists a differentiable embedding

$$\begin{array}{ccc} \psi: I^{k+1} \times M_\alpha & \longrightarrow & (I^{k+1} \times \tilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^{k+1} & \end{array}$$

so that

(1) $I^{k+1} \times M_{\alpha_1} \subset \mathrm{Int}\, \psi(I^{k+1} \times M_\alpha)$,

(2) $\psi(I^{k+1} \times M_\alpha) \subset \mathrm{Int}(I^{k+1} \times M_{\alpha_2})$,

(3) $\psi \mid I^k \times [0, 1/4] \times M_\alpha = \phi_1 \mid I^k \times [0, 1/4] \times M_\alpha$,

(4) $\psi \mid I^k \times [3/4, 1] \times M_\alpha = \phi_2 \cdot u \mid I^k \times [3/4, 1] \times M_\alpha$,

with

$$\begin{array}{ccc} u: I^k \times I_2 \times M_\alpha & \longrightarrow & I^k \times I_2 \times M_\alpha \\ & \searrow \quad \swarrow & \\ & I^k \times I_2 & \end{array}$$

The hypothesis is illustrated in Figure 5(1) and the conclusion in Figure 5(2).

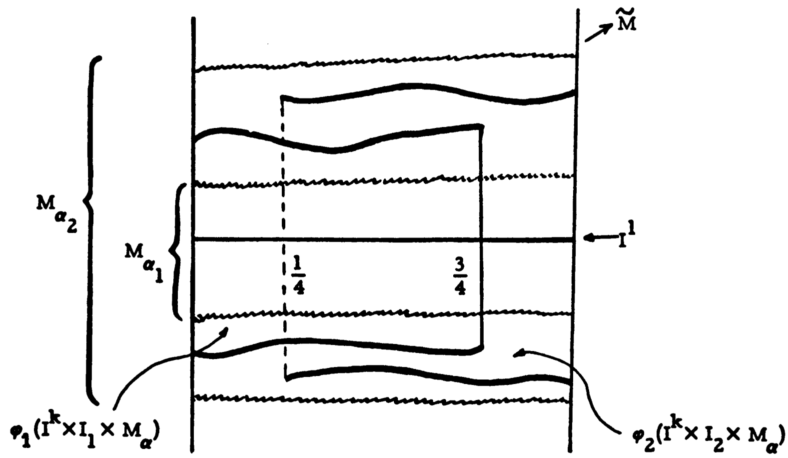


Figure 5(1)

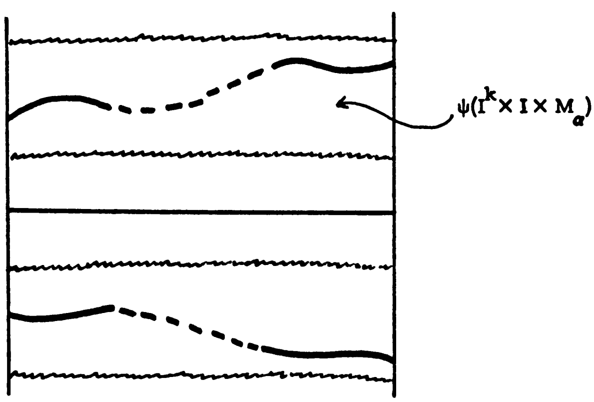


Figure 5(2)

Proof. We consider $\phi_{(1)}$ and $\phi_{(2)}$ and remark they verify the requirements of Proposition 2.3 for $I^{k+1} = I^k \times I_0$ and $y = x$, hence there exist $\alpha' > \alpha$, the differentiable embedding

$$\begin{array}{ccc} \tilde{\phi}_1 : I^k \times I_0 \times M_{\alpha'} & \longrightarrow & (I^k \times I_0 \times \tilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times I_0 & \end{array}$$

and the diffeomorphism

$$\begin{array}{ccc} h\colon I^k \times I_0 \times M_{\alpha'} & \longrightarrow & I^k \times I_0 \times M_\alpha \\ & \searrow \qquad \swarrow & \\ & I^k \times I_0 & \end{array}$$

so that $\tilde{\phi}_1 \mid I^k \times I_0 \times M_\alpha = \phi_{(1)}$ and $\tilde{\phi}_1 = \phi_{(2)} \cdot h$. Because we can extend h to a diffeomorphism

$$\begin{array}{ccc} \bar{h}\colon I^k \times [\tfrac{1}{4}, 1] \times M_{\alpha'} & \longrightarrow & I^k \times [\tfrac{1}{4}, 1] \times M_\alpha \\ & \searrow \qquad \swarrow & \\ & I^k \times [\tfrac{1}{4}, 1] & \end{array}$$

we can extend $\tilde{\phi}_1$ to a differentiable embedding

$$\begin{array}{ccc} \tilde{\tilde{\phi}}_1\colon I^k \times I_2 \times M_\alpha & \longrightarrow & (I^k \times I_2 \times \tilde{M})_\theta \\ & \searrow \qquad \swarrow & \\ & I^k \times I_2 & \end{array}$$

taking $\tilde{\tilde{\phi}}_1 = \phi_2 \cdot \bar{h}$. Clearly ϕ_1 and $\tilde{\tilde{\phi}}_1$ define a differentiable embedding

$$\begin{array}{ccc} \bar{\phi}_1\colon I^k \times I \times M_\alpha \cup I^k \times [\tfrac{1}{4}, 1] \times M_{\alpha'} & \longrightarrow & (I^k \times I \times \tilde{M})_\theta \\ & \searrow \qquad \swarrow & \\ & I^k \times I & \end{array}$$

We choose

$$\begin{array}{ccc} \xi\colon I^{k+1} \times M_\alpha & \longrightarrow & I^{k+1} \times M_{\alpha'} \\ & \searrow \qquad \swarrow & \\ & I^{k+1} & \end{array}$$

a differentiable embedding so that $\xi \mid I^k \times [0, 1/4] \times M_\alpha$ is the canonical inclusion, $\xi(I^{k+1} \times M_\alpha) \supseteq I^{k+1} \times M_{\alpha'}$, and

$$\begin{array}{ccc} \xi\colon I^k \times [3/4, 1] \times M_\alpha & \longrightarrow & I^k \times [3/4, 1] \times M_{\alpha'} \\ & \searrow \qquad \swarrow & \\ & I^k \times [3/4, 1] & \end{array}$$

is a diffeomorphism. We take $\psi = \bar{\phi}_1 \cdot \xi$ and check that the required conditions are satisfied.

In order to facilitate the reading of the proof of Theorem 2.2, we recall the topological ambient isotopy theorem (as in [18, p. 145]).

A.I.T. *Let C be a compact set of a topological manifold M without boundary, U a neighborhood of C in M and*

$$\begin{array}{ccc} h\colon I^k \times U & \longrightarrow & I^k \times M \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

a topological embedding. Then, there exists a homeomorphism

$$\begin{array}{ccc} g\colon I^k \times M & \longrightarrow & I^k \times M \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

such that $g_0 = \mathrm{id}$, g_t agrees with $g_{t'}$ outside of a compact set and $g_t \cdot h_0$ agrees with h_t on a neighborhood of C (by g_t, respectively h_t, we denote the restrictions of g, respectively h, to $\{t\} \times M$).

Corollary 2.5. *Let $\alpha_1 < \alpha^1 < \alpha_2 < \alpha^2 < \alpha_3$. There exists a differentiable embedding*

$$\begin{array}{ccc} \phi\colon I^k \times M_{\alpha^2} & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

such that

(1) $I^k \times M_{\alpha_i} \subset \phi(\mathrm{Int}\, I^k \times M_{\alpha^i})$, $i = 1, 2$,

(2) $\phi(I^k \times M_{\alpha^i}) \subset I^k \times \mathrm{Int}\, M_{\alpha_{i+1}}$, $i = 1, 2$,

(3) $\phi(\{y\} \times M_{\alpha^2}) \setminus (y \times \mathrm{Int}\, M_{\alpha_2})$ *is homeomorphic to* $\partial M_{\alpha_2} \times [0, 1]$, $(\{y\} \times M_{\alpha_2}) \setminus \phi(\{y\} \times \mathrm{Int}\, M_{\alpha^1})$ *is homeomorphic to* $\partial M_{\alpha_2} \times [0, 1]$ *for some* y.

Moreover, given $\phi_y\colon (\{y\} \times M_{\alpha^2}) \to (\{y\} \times \widetilde{M})_\theta$ compatible with (1), (2), *and* (3), *one can choose ϕ so that $\phi \mid (\{y\} \times M_{\alpha^2}) = \phi_y$.*

Remark. The topological ambient isotopy theorem implies (3) is true for any y as soon at it is true for one y. Theorem 2.2 and Corollary 2.5 will be proven simultaneously. For that, we denote by A_k the statement of Theorem 2.2 for fixed M, θ, k and arbitrary α_1, α^1, α_3 and by B_k the statement of Corollary 2.5 for fixed M, θ, k and arbitrary $\alpha_1, \alpha_2, \alpha_3, \alpha^1, \alpha^2$. We will show that $A_k \Rightarrow B_k$ and $B_k \Rightarrow A_{k+1}$.

$A_k \Rightarrow B_k$: Given $\alpha_1 < \alpha^1 < \alpha_2 < \alpha^2 < \alpha_3$ according to A_k, we can construct two differentiable embeddings

$$\begin{array}{ccc} \phi_1: I^k \times M_{\alpha^1} & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

and

$$\begin{array}{ccc} \phi_2: (I^k \times M_{\alpha^2}) & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

with $\alpha_1, \alpha^1, \alpha_2$ instead of $\alpha_1, \alpha^1, \alpha_3$ and $\alpha_2, \alpha^2, \alpha_3$ instead of $\alpha_1, \alpha^1, \alpha_3$ so that $\phi_2(\{0\} \times M_{\alpha^2}) \setminus \operatorname{Int} \phi_1(\{0\} \times M_{\alpha^1})$ is diffeomorphic to $\partial M_{\alpha^1} \times [\alpha^1, \alpha^2]$. ϕ_1 and $\phi_2 \circ k$, $k = \mathrm{id} \times$ diffeomorphism $M_{\alpha^1} \to M_{\alpha^2}$, verify the conditions of Proposition 2.3 for $x = 0$; hence we can construct

$$\begin{array}{ccc} \phi: I^k \times M_{\alpha^2} & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

with $\phi \mid I^k \times M_{\alpha^1} = \phi_1$ and $\phi(I^k \times M_{\alpha^2}) = \phi_2(I^k \times M_{\alpha^2})$ and B_k is proved.

$B_k \Rightarrow A_{k+1}$: Fix $\alpha_1 < \alpha^1 < \alpha_3$ and choose α_2, α^2 with $\alpha^1 < \alpha_2 < \alpha^2 < \alpha_3$. Consider $I^{k+1} = I^k \times I$ and for any $t \in I$ choose

$$\begin{array}{ccc} {}^t\phi: (I^k \times \{t\} \times M_{\alpha^2}) & \longrightarrow & (I^k \times \{t\} \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times \{t\} & \end{array}$$

differentiable embeddings according to B_k. As the first factor projection $p: (I^{k+1} \times \widetilde{M})_\theta \to I^{k+1}$ is a submersion we extend ${}^t\phi$ to

$$\begin{array}{ccc} {}^t\widetilde{\phi}: I^k \times [t - {}^t\epsilon, t + {}^t\epsilon] \times M_{\alpha^2} & \longrightarrow & (I^k \times [t - {}^t\epsilon, t + {}^t\epsilon] \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times [t - {}^t\epsilon, t + {}^t\epsilon] & \end{array}$$

a differentiable embedding. Using the compactness of $I = [0, 1]$, we find an N and $0 < t_0 < t_1 < \cdots < t_N = 1$ so that ${}^{t_i}\epsilon = \epsilon_i$ has the property $[t_i - \epsilon_i, t_i + \epsilon_i] \supset [t_{i-1}, t_{i+1}]$.

We will define inductively

$$\begin{array}{ccc} {}_i\phi\colon M_{\alpha^1} \times I^k \times [0, t_i] & \longrightarrow & (M \times I^k \times [0, t_i])_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times [0, t_i] & \end{array}$$

so that ${}_i\phi$ satisfies the conditions required by A_{k+1} and ${}_i\phi | I^k \times [t_i - \epsilon, t_i] \times M_{\alpha^1} = \phi_i \circ h | I^k \times [t_i - \epsilon, t_i] \times M_{\alpha^1}$, ϵ small enough, where $\phi_i = {}^t\phi$ with $t = t_i$ and

$$\begin{array}{ccc} h\colon I^k \times [t_{i-1} - \epsilon, t_i] \times M_{\alpha^1} & \longrightarrow & (I^k \times [t_{i-1} - \epsilon, t_i] \times M)_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times [t_{i-1} - \epsilon, t_i] & \end{array}$$

is a diffeomorphism. Assume ${}_i\phi$ defined; we wish to define ${}_{i+1}\phi$ on $I^k \times [0, t_{i+1}] \times M_{\alpha^1}$. We notice that ${}_i\phi$ and $\phi_{i+1} \circ (\mathrm{id} \times S)$ (instead of $\phi_{(1)}$ and $\phi_{(2)}$) satisfy the conditions of Proposition 2.4 (with $t_i - \epsilon$ and t_i instead of 1/4 and 3/4) with S any diffeomorphism from M_{α^1} to M_{α^2}. Hence one can construct

$$\begin{array}{ccc} {}_{i+1}\phi'\colon I^k \times [0, t_{i+1}] \times M_{\alpha^1} & \longrightarrow & (I^k \times [0, t_{i+1}] \times M)_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times [0, t_{i+1}] & \end{array}$$

with ${}_{i+1}\phi'$ equal to ${}_i\phi$ on $I^k \times [0, t_i - \epsilon] \times M_{\alpha^1}$ and ${}_{i+1}\phi'$ equal to $\phi_{i+1} \cdot u$ (see Proposition 2.4) on $I^k \times [t_i, t_{i+1}] \times M_{\alpha^1}$. Figure 6(1) and 6(2) suggest the construction of ${}_{i+1}\phi'$.

Figure 6(1)

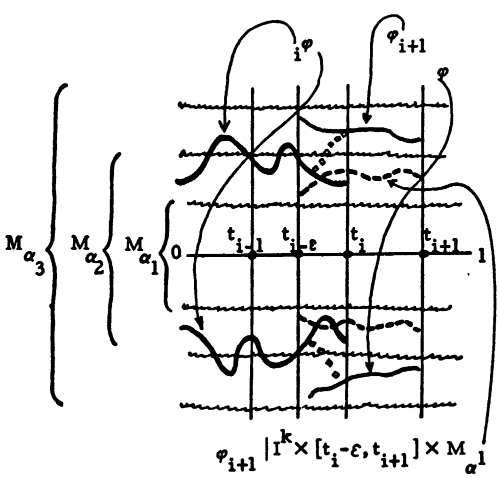
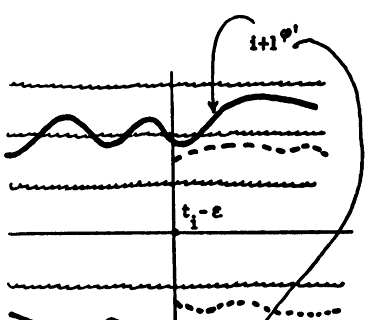


Figure 6(2)



We change ${}_{i+1}\phi'$ into ${}_{i+1}\phi$ taking ${}_{i+1}\phi = {}_{i+1}\phi' \cdot \eta$ and choosing a differentiable embedding

$$\begin{array}{ccc} \eta: I^k \times [0, t_{i+1}] \times M_{\alpha^1} & \longrightarrow & I^k \times [0, t_{i+1}] \times M_{\alpha^1} \\ & \searrow \quad \swarrow & \\ & I^k \times [0, t_{i+1}] & \end{array}$$

so that $\eta | I^k \times [0, t_i] \times M_{\alpha^1} \equiv \mathrm{id}$, ${}_{i+1}\phi' \cdot \eta(I^k \times [t_{i+1} - \epsilon, t_{i+1}] \times M_{\alpha^1}) \equiv \phi_{i+1}(I^k \times [t_{i+1} - \epsilon, t_{i+1}] \times M_{\alpha^1})$ as indicated in Figure 6(3).

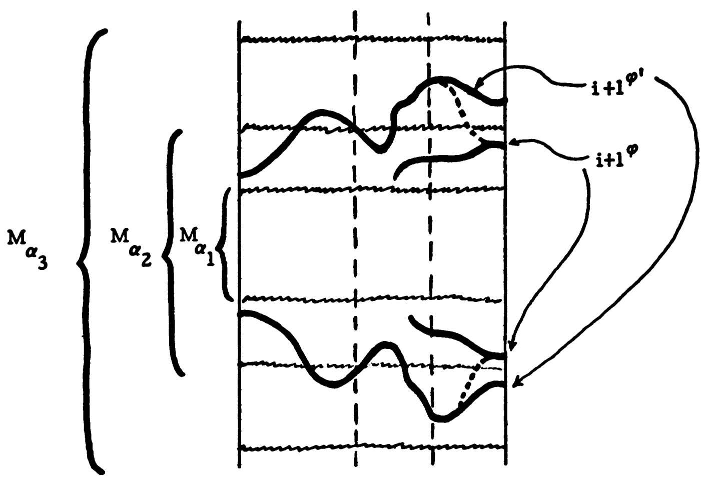


Figure 6(3)

It is clear that we can start with

$$\begin{array}{ccc} {}^0\phi: I^k \times \{0\} \times M_{\alpha^2} & \longrightarrow & (I^k \times \{0\} \times \tilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k \times \{0\} & \end{array}$$

with ${}^0\phi | I^{k'} \times \{0\} \times M_{\alpha^1} = \psi$ as soon as ${}^0\phi$ is compatible with (1) and (2). Because A_0 is true, as we have shown at the beginning of this chapter, by induction one obtains "A_k is true" for any k. We need $n \neq 4, 5$ or $n = 5$ and $\partial M^5 = S^4$ to conclude from "$L = \phi(\{x\} \times M_\beta) \backslash \phi(\mathrm{Int}(\{x\} \times M_\alpha))$ is homeomorphic to $\partial M_\beta \times [\beta, \alpha]$" that L is also diffeomorphic to $\partial M_\beta \times [\beta, \alpha]$. This is always the case if L is pd homeomorphic to $\partial |M_\beta| \times [\beta, \alpha]$ (via the classical smoothing theory). Q.E.D.

The condition for $n = 5$, that $\partial M = S^4$ is not essential, but the argument requires a modification which we sketch: Let I_i, $i = 0, 1, 2$, be as in 2.4 and fix $x \in I^k$. Suppose we are given differentiable embeddings

$\phi_1\colon I_1 \times I^k \times M_\alpha \to (I_1 \times I^k \times \widetilde{M})_\theta$, and $\phi_2\colon I_2 \times I^k \times (\operatorname{Int} M_\beta)_\theta \to (I_2 \times I^k \times M_\beta)_\theta$,

commuting with projection, where $(\operatorname{Int} M_\beta)_\theta = (1 \times x \times \operatorname{Int} M_\beta)_\theta$ and $\phi_2|(1 \times x \times \operatorname{Int} M_\beta) =$ inclusion. Assume further

(1) $\phi_i(I_0 \times I^k \times \operatorname{Int} M_\alpha) \supset I^k \times I_0 \times M_{\alpha_1}$, $\alpha_1 < \alpha$, $i = 1, 2$,

(2) $\phi_1(I_0 \times I^k \times M_\alpha) \subset I_0 \times I^k \times \operatorname{Int} M_{\alpha_2}$, $\phi_2(I_0 \times I^k \times \operatorname{Int} M_\beta) \subset I_0 \times I^k \times \operatorname{Int} M_{\alpha_2}$, $\alpha_2 > \alpha$,

(3) $\phi_1(I_0 \times I^k \times M_\alpha) \subset \operatorname{Int} \phi_2(I_0 \times I^k \times M_\alpha)$,

(4) $\phi_1\colon \{0 \times 0\} \times \partial M_\alpha \to \{0 \times 0\} \times (\widetilde{M} - M_{\alpha_1})$ is a homotopy equivalence,

(5) $\phi_2(I_0 \times I^k \times M_{\alpha_0}) \subset I_0 \times I^k \times M_\alpha$, $\alpha_0 < \alpha_1$.

Then by the lemma of Chapter 7 of [17] quoted below, there exists for any $t \in I_0$ a differentiable embedding $\lambda\colon M_\alpha \to (\operatorname{Int} M_\beta)_\theta$ such that $\lambda(M_\alpha) \supset M_\alpha$ and $\phi_2 \circ (1 \times \lambda)(\{t \times x\} \times M_\alpha) - \phi_1(\{t \times x\} \times \operatorname{Int} M_\alpha)$ is diffeomorphic to $\partial M_\alpha \times [0, 1]$. (Note $\phi_2^{-1}\phi_1(\{t \times x\} \times \partial M_\alpha) \subset \{t \times x\} \times (\operatorname{Int} M_\beta - \operatorname{Int} M_{\alpha_0})_\theta$.) Hence if we replace ϕ_2 by $\phi_2 \circ (1 \times \lambda)$ and set $y = t \times x$, all the conditions of 2.4 are satisfied.

Now replace homeomorphic in (2) of Theorem 2.2 (3 of Corollary 3.5) by diffeomorphic. Then in the proof of $B_k \Rightarrow A_{k+1}$ we take ${}^t\phi$ to be an embedding ${}^t\phi\colon \{t\} \times I^k \times (M_{\alpha^2})_\theta \to (\{t\} \times I^k \times M)_\theta$, where $(M_{\alpha^2})_\theta = (\{t \times x\} \times M_{\alpha^2})_\theta$, some $x \in I^k$, and ${}^t\phi|(\{t \times x\} \times M_{\alpha^2})$ is inclusion. We change ${}^t\widetilde{\phi}$ similarly to obtain the conditions above.

Lemma (Chapter 7 of [18]). *Let M^n, $n \geq 5$, be a smooth manifold. Let M be homeomorphic to $K \times R$, where K is a closed connected topological manifold. If there is a smooth submanifold $N^{n-1} \subset M$, such that N is a deformation retract of M, then M is diffeomorphic to $N \times R$, with N corresponding to $N \times 0$.*

Theorem 2.2′. *Under the hypothesis of Theorem 2.2, there exists a diffeomorphism*

$$\begin{array}{ccc} \phi\colon I^k \times \widetilde{M} & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

Proof. By applying Proposition 2.3 inductively where $\alpha = 1, 2, 3, \cdots$ and taking $\psi(I^k \times M_\alpha) \supset I^k \times M_\alpha$ (which we can do by Theorem 2.2), we can extend ϕ in Theorem 2.2 step-by-step to the required diffeomorphism.

We can remove the restriction that $\widetilde{M} = M \cup \partial M \times [0, \infty)$ and consider the case of any connected open smooth manifold M. In fact, $M = \bigcup_{n=1}^\infty M_n$, where M_n is a compact connected smooth manifold with boundary with $M_n \subset \operatorname{Int} M_{n+1}$. Suppose that by Theorem 2.2$'$, we have obtained a diffeomorphism $\phi_n: I^k \times \tilde{M}_n \to (I^k \times \tilde{M}_n)_\theta$, where $\tilde{M}_n$ is a collar neighborhood of M_n in Int M_{n+1}. Then by applying relative smoothing theory, for any $t \in I^k$ we can extend $\phi_n | t \times (M_n)_\alpha$ to a diffeomorphism of $t \times \tilde{M}_{n+1}$ onto $(t \times \tilde{M}_{n+1})_\theta$. Now the construction of Proposition 2.3 and the proof of Theorems 2.2 and 2.2$'$ result in a diffeomorphism $\phi_{n+1}: I^k \times \tilde{M}_{n+1} \to (I^k \times \tilde{M}_{n+1})_\theta$ which agrees with ϕ_n on $I^k \times M_n$. Hence by induction on n we get:

Theorem 2.2$''$. *Let M be any smooth connected open manifold. Let θ be a smoothing of $I^k \times M$ such that the projection $p: (I^k \times M)_\theta \to I^k$ is a smooth submersion, and the smoothing $((0) \times M)_\theta$ is concordant to the given smooth structure on M. Then there exists a diffeomorphism*

$$\begin{array}{ccc} \phi: I^k \times M & \to & (I^k \times M)_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

From Theorem 2.2$''$ we get

Theorem 2.6. *Let E^{n+k} and B^k be smooth manifolds and $p: E \to B$ a smooth submersion so that*

(1) *p is a topologically locally trivial fibration,*

(2) *$n \neq 4$.*

Then p is a smooth locally trivial fibration.

Remark 2.6$'$. If p is a proper map, the conclusion holds without conditions (1) and (2).

Remark 2.6$''$. If we replace (1) by

(1$'$) p is a pd locally trivial fibration,

then (2) is superfluous.

Proof. It is clearly sufficient to prove 2.6 when $B^k = I^k$. But this follows from 2.2$''$.

3. The homotopy groups of $(\mathrm{E}^t_-, \mathrm{E}^d_-)$ and $(\mathrm{E}^{pd}_-, \mathrm{E}^d_-)$. The aim of this chapter is to prove Corollary 3.2.

Theorem 3.1. (a) *The natural inclusion*

$$(\mathrm{Im}^d(M^n, N^n), \mathrm{E}^d(M^n, N^n)) \hookrightarrow (\mathrm{Im}^t(M^n, N^n), \mathrm{E}^t(M^n, N^n))$$

induces, for any basepoint (i.e. 0-simplex) $x \in \mathrm{Im}^d(M^n, N^n)$, a bijective map

$$\pi_i(\mathrm{Im}^d(M^n, N^n), \mathrm{E}^d(M^n, N^n); x) \to \pi_i(\mathrm{Im}^t(M^n, N^n), \mathrm{E}^t(M^n, N^n); x)$$

(if $n \neq 4$); (if $i \geq 2$ then x has to be in $\mathrm{E}^d(M^n, N^n)$).

(b) *The injective semisimplicial map*

$$t^*\colon (\mathrm{Im}^d(M^n, N^n), \mathrm{E}^d(M^n, N^n)) \hookrightarrow (\mathrm{Im}^{pd}(|M^n|, N^n), \mathrm{E}^{pd}(|M^n|, N^n))$$

induces for any point (i.e. 0-simplex) $x \in \mathrm{Im}^d(M^n, N^n)$, *a bijective map*

$$\pi_i(\mathrm{Im}^d(M^n, N^n), \mathrm{E}^d(M^n, N^n); x) \longrightarrow \pi_i(\mathrm{Im}^{pd}(M^n, N^n), \mathrm{E}^{pd}(M^n, N^n); t^*x)$$

(if $i \geq 2$, x *has to be in* E^d*).*

Corollary 3.2. (a) *The natural inclusion of pairs*

$$(\mathrm{E}^t(M^n, N^n), \mathrm{E}^d(M^n, N^n)) \hookrightarrow (\mathrm{Im}^t(M^n, N^n), \mathrm{Im}^d(M^n, N^n))$$

induces for any basepoint (i.e., 0-simplex) $x \in \mathrm{E}^d(M^n, N^n)$ *a bijective map (for all* $i \geq 1$*)*

$$\pi_i(\mathrm{E}^t(M^n, N^n), \mathrm{E}^d(M^n, N^n); x) \longrightarrow \pi_i(\mathrm{Im}^t(M^n, N^n), \mathrm{Im}^d(M^n, N^n); x)$$

(if $n \neq 4$*).*

(b) *The natural inclusion of pairs*

$$(\mathrm{E}^{pd}(|M^n|, N^n), t^*\mathrm{E}^d(M^n, N^n)) \hookrightarrow (\mathrm{Im}^{pd}(|M^n|, N^n), t^*\mathrm{Im}^d(M^n, N^n))$$

induces for any basepoint (i.e. 0-simplex) $x \in t^*\mathrm{E}^d(M^n, N^n)$ *a bijective map for* $i \geq 1$

$$\pi_i(\mathrm{E}^{pd}(|M^n|, N^n), t^*(\mathrm{E}^d(M^n, N^n), x) \longrightarrow \pi_i(\mathrm{Im}^{pd}(|M^n|, N^n), t^*\mathrm{Im}^d(M^n, N^n, x).$$

Standard exact sequence-arguments with special case for $i = 1$ show that Theorem 3.1 implies Corollary 3.2.

The proof of Theorem 3.1 is a formal consequence of Theorem 2.2 and of the ambient isotopy theorem (denoted by A. I. T.) in the topological case and pd case. The topological A. I. T. has already been stated in §2 and the A. I. T. in the pd case follows combining the A. I. T. in the pl case ([14, p. 154] and the Whitehead triangulation as in [21] or [26]).

We will prove only Theorem 3.1(a); the proof of (b) being almost the same with the only difference that one uses the A. I. T. in the pd case instead of the A. I. T. in the topological case.

Proof of surjectivity. We can represent an element of $\pi_k(\mathrm{Im}^t(\ ), \mathrm{E}^t(\ ), x)$ by a topological immersion

$$\begin{array}{ccc} \sigma\colon \Delta[k] \times \widetilde{M} & \longrightarrow & \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

so that σ_t, the restriction of σ to $\{t\} \times \widetilde{M}$ is a topological embedding for any

$t \in \mathring{\Delta}[k]$. Obviously one can extend σ to a topological immersion

$$\begin{array}{ccc} \bar{\sigma}: I^k \times \widetilde{M} & \longrightarrow & I^k \times N \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

with $\bar{\sigma}|\Delta[k] \times \widetilde{M} = \sigma$ and $\bar{\sigma}_t$ a topological embedding for any $t \in I^k \setminus \operatorname{Int} \Delta[k]$. One considers the following commutative diagram

$$\begin{array}{ccccc} I^k \times \widetilde{M} & \xrightarrow{\text{id}} & (I^k \times \widetilde{M})_\theta & \xrightarrow{\bar{\sigma}} & I^k \times N \\ & \searrow & \downarrow & \swarrow & \\ & & I^k & & \end{array}$$

where θ represents the topological fibrewise smoothing induced on $I^k \times \widetilde{M}$ by σ. $\text{id}|\{0\} \times \widetilde{M}$ is obviously a differentiable embedding because $\sigma|\{0\} \times \widetilde{M}$ is. According to Theorem 2.2 we can construct a differentiable embedding

$$\begin{array}{ccc} \sigma: I^k \times M_{\alpha_1} & \longrightarrow & (I^k \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

with $\phi_0 = \phi|\{0\} \times M_{\alpha_1} = \text{id}$. Applying the A. I. T., we can find

$$\begin{array}{ccc} h: I^k \times \widetilde{M} & \longrightarrow & I^k \times \widetilde{M} \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

with $h \cdot (\text{id} \times \phi_0)|I^k \times M_{\alpha_2} = \phi$ for any $\alpha_2 < \alpha_1$, and because $h_0 = \text{id}$ we can find a family of isotopies

$$\begin{array}{ccc} h^s: I^k \times \widetilde{M} & \longrightarrow & I^k \times \widetilde{M} \\ & \searrow \quad \swarrow & \\ & I^k & \end{array}$$

$s \in [0, 1]$, depending continuously on s with $h_0^s = \text{id}$, $h^0 = \text{id}$ and $h^1 = h$.

We consider then $\sigma^s = \sigma \cdot h^s|\Delta[k] \times M_{\alpha_2}$

$$\begin{array}{ccc} \sigma^s: \Delta[k] \times M_{\alpha_2} & \longrightarrow & \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

It is clear that σ_t^s is an embedding for any t for which $\bar{\sigma}_t$ was, because h_t^s and σ_t are embeddings, $\sigma_0^s = \sigma_0^0$, $\sigma^0 = \sigma$, and σ^1 is a differentiable immersion. Hence

σ^1 represents the same element in $\pi_i(\mathrm{Im}^t(\ ), \mathrm{E}^t(\ ), x)$ as σ^0 and at the same time it represents an element in $\pi_i(\mathrm{Im}^d(\ ), \mathrm{E}^d(\ ), x)$.

Proof of injectivity. For simplicity we will prove the injectivity only in the case $k \geq 2$ when $\pi_k(\mathrm{Im}(\ ), \mathrm{E}(\ ), x)$ is a group; the reader could complete the proof for $k = 1$, using the Theorem 2.2(b) and the remark to Theorem 2.2". Consider

$$\begin{array}{ccc} \sigma\colon \Delta[k] \times Q & \longrightarrow & \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

a differentiable immersion representing an element in $\pi_k(\mathrm{Im}^d(\ ), \mathrm{E}^d(\ ), x)$, i.e. Q is a neighborhood of M in $\widetilde{M}$ and $\sigma_t\colon Q \to N$ is a differentiable embedding for $t \in \dot{\Delta}[k]$. Assume σ represents in $\pi_k(\mathrm{Im}^t(\ ), \mathrm{E}^t(\ ), x)$ the trivial element. This means there exist

$$\begin{array}{ccc} \bar{\sigma}\colon I \times \Delta[k] \times Q' & \longrightarrow & I \times \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & I \times \Delta[k] & \end{array}$$

a topological immersion, with $\bar{\sigma}_t\colon Q' \to N$ an embedding for any $t \in I \times \dot{\Delta}[k] \cup \{1\} \times \Delta[k]$, and Q' a neighborhood of M in $\widetilde{M}$ so that $\bar{\sigma}|\{0\} \times \Delta[k] \times Q'' = \sigma|\Delta[k] \times Q''$, Q'' being a neighborhood of M in $Q \cap Q'$. One can always assume Q'' diffeomorphic to M. One chooses

$$\begin{array}{ccc} \bar{\bar{\sigma}}\colon I^{k+1} \times \widetilde{M} & \longrightarrow & I^{k+1} \times N \\ & \searrow \quad \swarrow & \\ & I^{k+1} & \end{array}$$

so that $\bar{\bar{\sigma}}|I \times \Delta[k] = \bar{\sigma}$ and $\bar{\bar{\sigma}}|\{0\} \times I^k$ is a differentiable immersion and $\bar{\bar{\sigma}}|I \times (I^k - \mathrm{Int}\,\dot{\Delta}[k])$ is an embedding.

One considers

$$\begin{array}{ccccc} I^{k+1} \times \widetilde{M} & \xrightarrow{\ \mathrm{id}\ } & (\dot{I}^{k+1} \times \widetilde{M})_\theta & \xrightarrow{\ \bar{\bar{\sigma}}\ } & I^{k+1} \times N \\ & \searrow & \downarrow & \swarrow & \\ & & I^{k+1} & & \end{array}$$

with θ the topological fibrewise smoothing induced by $\bar{\bar{\sigma}}$; clearly $\mathrm{id}|\{0\} \times M$ is differentiable of maximal rank, and applying Theorem 2.2 one can find

$$\begin{array}{ccc} l\colon I^{k+1} \times M_\alpha & \longrightarrow & (I^{k+1} \times \widetilde{M})_\theta \\ & \searrow \quad \swarrow & \\ & I^{k+1} & \end{array}$$

a differentiable embedding such that l and "id" agree on $\{0\} \times I^k \times M_\alpha$. We take

$$\begin{array}{ccc} \tilde{\sigma} = \bar{\sigma} \cdot l\colon \Delta[k] \times M_\alpha & \longrightarrow & \Delta[k] \times N \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

which is a differentiable immersion and notice that $\tilde{\sigma}_{(t)}\colon M_\alpha \to N$ is a differentiable embedding for any $t \in I \times \dot{\Delta}[k] \cup \{1\} \times \Delta[k]$, because $\bar{\sigma}_{(t)}$ and $l_{(t)}$ are imbeddings. The existence of $\tilde{\sigma}$ proves the triviality of the element represented by σ in $\pi_k(\mathrm{Im}^d(\ ), \mathrm{E}^d(\ ), x)$. Q.E.D.

4. The main theorems. As in the previous sections, let M^n be a compact differentiable manifold with or without boundary and $t\colon |M^n| \to M^n$ a differentiable triangulation. If $\partial M^n \neq \emptyset$ this means $t\,|\partial|M^n| \to \partial M^n$ is also a differentiable triangulation. Assume now N^p is a compact differentiable submanifold of M^n. (In this case the triangulation t is assumed to have the property that $t^{-1}(N^p)$ is a pl submanifold of $|M^n|$.) In what follows N will be either ∂M^n or an $(n-1)$ differentiable submanifold of ∂M^n or an n-dimensional submanifold of M^n so that $\partial M^n \subset N$. In this section we will prove the Theorems 4.2, 4.5 which will become the key facts of our study. To simplify the writing we let $S\,\mathrm{Homeo}(M^n, N)/S^d\mathrm{Diff}(M^n, N) = TD_N^M$ and $\mathrm{PD}(M^n, t, N)/S^d\mathrm{Diff}(M^n, N) = PlD_N^M$.

Definition 4.1. $f\colon X \to Y$ is called an *IHE*-map if it induces an injective correspondence for connected components and an homotopy equivalence on any connected component.

Theorem 4.2. *If* $\dim N \geq \dim M - 1$ *there exist basepoint preserving maps* $i^t\colon \{TD_N^M\} \to \Gamma^N(P^t, {}^{\partial}P^t; s)$ *and* $i^{pl}\colon \{PlD_N^M\} \to \Gamma^N(P^{pl}, {}^{\partial}P^{pl}; s)$ *so that*

(1^t) *if* $n \neq 4$ *then* $i^t\colon \{TD_{\partial M}^{M^n}\} \to \Gamma^{\partial M}(P^t, s) \equiv \Gamma^{\partial M}(P^t, {}^{\partial}P^t, s)$ *is an IHE map;*

(2^t) *if* $\partial M = \emptyset$ *and* $n \neq 4$ *then* $i^t\colon \{TD^{M^n}\} \to \Gamma(P^t, s)$ *is an IHE map;*

(3^t) *if* N^{n-1} *is a compact submanifold of* ∂M^n *and* $n \neq 4, 5$, *then* $i^t\colon \{TD_N^{M^n}\} \to \Gamma^N(P^t, {}^{\partial}P^t, s)$ *is an IHE-map;*

(4^t) *if* L^n *is a compact differentiable submanifold contained in* $M^n \setminus N$ *then the diagram*

$$\begin{array}{ccc} \{TD_{\partial L^n}^{L^n}\} & \longrightarrow & \Gamma^{\partial L^n}(P_L^t, S) \\ \Big\downarrow \gamma_1^t & & \Big\downarrow \gamma_2^t \\ \{TD_N^M\} & \longrightarrow & \Gamma^N(P^t, {}^{\partial}P^t, s) \end{array}$$

is homotopy commutative, where γ_1^t, *respectively* γ_2^t, *are defined extending by id outside* L, *respectively extending by* s *outside* L.

(1^{pl}): *$i^{pl}: \{PlD^M_{\partial M}\} \to \Gamma^{\partial M}(P^{pl}, s)$ is an IHE map,*

(2^{pl}): *if $\partial M = \emptyset$, $i^{pl}: \{PlD^M\} \to \Gamma(P^{pl}, s)$ is an IHE map,*

(3^{pl}): *if N^{n-1} is a compact submanifold of ∂M^n then $i^{pl}: \{PlD^M_N\} \to \Gamma(P^{pl}, {}^\partial P^{pl}, s)$ is an IHE-map,*

(4^{pl}): *if L^n is a compact differentiable submanifold contained in $M^n \setminus N$ then the diagram*

$$\begin{array}{ccc} \{PlD^{L^n}_{\partial L^n}\} & \longrightarrow & \Gamma^{\partial L^n}(P^{pl}_L, s) \\ \Big\downarrow \gamma_1^{pl} & & \Big\downarrow \gamma_2^{pl} \\ \{PlD^M_N\} & \longrightarrow & \Gamma^N(P^{pl}_M, \partial P^{pl}_M, s) \end{array}$$

with γ_1^{pl} and γ_2^{pl} defined in the same way as γ_1^t and γ_2^t.

Firstly, we notice that according to Proposition 1.6 it suffices to prove Theorem 4.2 for $\mathbf{TD}^M_N \doteq \mathrm{Homeo}(M^n, N)/\mathrm{Diff}(M^n, N)$ instead of TD^M_N and for $\mathbf{P}l\mathbf{D}^M_N = \mathrm{PD}(M^n, N, t)/\mathrm{Diff}(M^n, N)$ instead of PlD^M_N.

It is also easy to identify ${}^\partial\mathbf{R}^t(M^n, N; M^n)/{}^\partial\mathbf{R}^d(M^n, N; M^n)$ in a natural way to a union of connected components of $S\Gamma^N(P^t_M, P^t, s)$ and ${}^\partial\mathbf{R}^{pd}(|M^n|, |N^n|; M^n)/{}^\partial\mathbf{R}^d(M^n, N^n, M^n)$ to a union of connected components of $S\Gamma^N(P^{pl}_M, {}^\partial P^{pl}_M, s)$ (up to a homotopy equivalence).

To see this, let TM be the tangent vector bundle of M and $|TM|$ its underlying topological R^n-bundle.

1. $\underline{\mathbf{R}}^t(M, M)/(\underline{\mathbf{R}}^d(M, M))$ is in 1-1 correspondence with vector bundle structures on $|TM|$ which are equivalent to TM.

In fact, if $f\colon |TM| \to |TM|$ is an R^n-bundle representation over the identity, i.e., an R^n-bundle equivalence, there is a unique vector bundle structure $|TM|_f$ on $|TM|$ such that f becomes a vector bundle equivalence. Further, if $g\colon TM \to TM$ is a vector bundle equivalence, then $|TM|_{g\circ f}$ and $|TM|_f$ define the same vector bundle structure on $|TM|$; i.e., $\mathrm{id}_{|TM|}\colon |TM|_{g\circ f} \to |TM|_f$ is a vector bundle equivalence.

Conversely, given a vector bundle structure $|TM|_a$ on $|TM|$, vector bundle equivalent to TM; i.e., there exists $f\colon |TM|_a \simeq TM$; then considered as an R^n-bundle equivalence, $f\colon |TM| \to |TM|$ is well-defined up to a vector bundle equivalence $g\colon TM \to TM$ by a.

2. Vector bundle structures on $|TM|$ are in 1-1 correspondence with sections of P^t_M (see Steenrod, *Fibre bundles*).

Now let $\Gamma(P^t_M)$ be the space of sections of P^t_M and $\Gamma_0(P^t_M)$ the components corresponding to vector bundle structures equivalent as vector bundles to TM. Let $S\Gamma_0(P^t_M)$ be the singular complex. A k-simplex of $\underline{\mathrm{R}}^t(M, M)$ defines a cross-section of $\Delta_k \times P^t_M$; i.e., a k-simplex of $S\Gamma_0(P^t_M)$, which is unchanged by composition with a k-simplex of $\underline{\mathrm{R}}^d(M, M)$. Conversely, a k-simplex of $S\Gamma_0(P^t_M)$ defines a vector bundle structure on $\Delta_k \times |TM|$ and hence a k-simplex of $\underline{\mathrm{R}}^t(M, M)$ mod $\underline{\mathrm{R}}^d(M, M)$ by (1) and (2).

Proposition 1.4 gives the semisimplicial maps

$$\delta^{t,d}\colon \mathrm{Homeo}(M^n, N)/\mathrm{Diff}(M^n, N) \to {}^{\partial}\mathrm{R}^t(M^n, N; M^n)/{}^{\partial}\mathrm{R}^d(M, N; M^n)$$

and

$$\delta^{pd,d}\colon \mathrm{PD}(M^n, N, t)/\mathrm{Diff}(M^n, N) \to {}^{\partial}\mathrm{R}^{pd}(|M^n|, |N|; |M^n|)/{}^{\partial}\mathrm{R}^d(M^n, N; M^n).$$

If we denote by $u^{t,d}$ and $u^{pd,d}$ the natural maps (see Proposition 1.6, §1) $u^{t,d}\colon \mathrm{TD}^M_N \to TD^M_N$ and $u^{pd,d}\colon \mathrm{P}l\mathrm{D}^M_N \to PlD^M_N$ and by $v^{t,d}$ respectively $v^{pd,d}$ their homotopy inverses (recall dim $N \geq$ dim $M - 1$) then we put $i^t = \delta^{t,d} \cdot v^{t,d}$ and $i^{pl} = \delta^{pd,d} \cdot v^{pd,d}$. Having in mind Proposition 1.4 (with all points 1, 2, 3, 4) we obtain easily the points (1^t), (1^{pl}), (2^t), (2^{pl}), (3^t), (3^{pl}) as soon as we have proved Proposition 4.3 below. (To show injectivity on connected components we use standard smoothing theory as presented in [18].) The points (4^t) and (4^{pl}) follow from the naturality of $\delta^{t,d}$ and $\delta^{pd,d}$.

Proposition 4.3. (1^t) *If* $N = \partial M^n \neq \emptyset$, $n \neq 4$, *then for any basepoint* $x \in \mathrm{Diff}(M^n, N)$ *the map of pairs*

$$\delta^t\colon (\mathrm{Homeo}(M^n, \partial M^n), \theta\, \mathrm{Diff}(M^n, \partial M^n)) \to (\mathrm{R}^t(M^n, \partial M^n, M^n), \theta \mathrm{R}^d(M^n, \partial M^n, M^n))$$

induces an isomorphism for homotopy groups (sets) π_i, $i \geq 1$.

(1^{pl}) *If* $N = \partial M^n \neq \emptyset$, *then for any basepoint* $x \in \mathrm{Diff}(M^n, N)$ *the map of pairs*

$$\delta^{pd}\colon (\mathrm{PD}(M^n, \partial M^n), t_* \mathrm{Diff}(M^n, \partial M^n)) \to (\mathrm{R}^{pd}(|M^n|, \partial|M^n|, M^n), t_*\mathrm{R}^d(M^n, \partial M^n, M^n))$$

induces an isomorphism for homotopy groups (sets) π_i, $i \geq 1$.

(2^t) *If* $N = \partial M^n = \emptyset$, $n \neq 4$ *then for any basepoint* $x \in \mathrm{Diff}(M^n)$ *the map of pairs*

$$\delta^t\colon (\mathrm{Homeo}(M^n), \theta\, \mathrm{Diff}(M^n)) \to (\mathrm{R}^t(M^n, M^n), \theta \mathrm{R}^d(M^n, M^n))$$

induces an isomorphism for homotopy groups (sets) π_i, $i \geq 1$.

(2^{pl}) *If* $N = \partial M^n = \emptyset$, *for any basepoint* $x \in \mathrm{Diff}(M^n)$ *the map of pairs*

$$\delta^{pd}\colon (\mathrm{PD}(M^n, t), \theta\, \mathrm{Diff}(M^n)) \to (\mathrm{R}^{pd}(|M^n|, M^n), \theta \mathrm{R}^d(M^n, M^n))$$

induces an isomorphism for homotopy groups (sets) π_i, $i \geq 1$.

(3^t) *If* $\partial M^n \neq \emptyset$, N^{n-1} *is a compact differentiable submanifold of* ∂M^n, $n \neq 4, 5$, *then for any basepoint* $x \in \mathrm{Diff}(M^n)$ *the map of pairs*

$$\delta^t: (\mathrm{Homeo}(M^n, N), \theta\, \mathrm{Diff}(M^n, N)) \to ({}^{\partial}\mathrm{R}^t(M^n, N, M^n), \theta^{\partial}\mathrm{R}^d(M^n, N, M^n))$$

induces an isomorphism for homotopy groups (sets) π_i, $i \geq 1$.

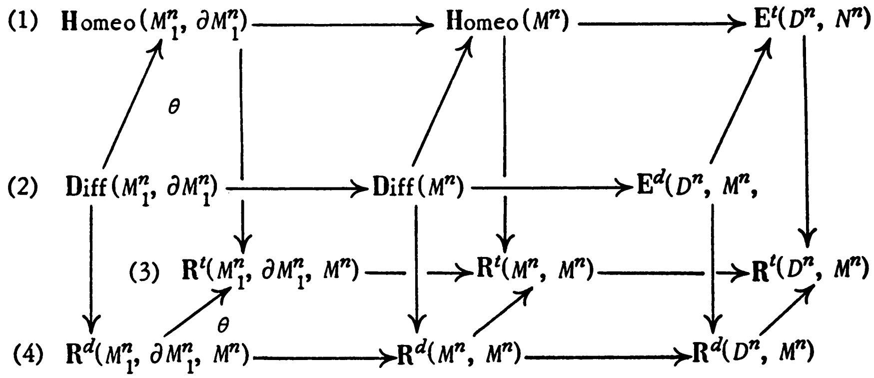
(3^{pl}) *If* $\partial M^n \neq \emptyset$, N^{n-1} *and* $x \in \mathrm{Diff}(M^n)$ *as in* (3^t),

$$\delta^{pd}: (\mathrm{PD}(M, N; t), t_* \mathrm{Diff}(M, N)) \to ({}^{\partial}\mathrm{R}^{pd}(|M|, |N|; M), t_* \mathrm{R}^d(M, N; M))$$

induces an isomorphism for homotopy groups (sets), π_i, $i \geq 1$.

Proof. (1^t) and (1^{pl}) follow by combining Theorem 1.1, Corollary 3.2 and the remark $\mathrm{Diff}(M^n, P^n) \subseteq \mathrm{Diff}(M^n, \partial M^n)$. $\mathrm{Homeo}(M^n, P^n) \subseteq \mathrm{Homeo}(M^n, \partial M^n)$ and $\mathrm{PD}(M^n, t, P^n) \subseteq \mathrm{PD}(M^n, t, \partial M^n)$ are homotopy equivalences which follow, for instance, from Propositions 1.5 and 1.6.

To prove (2^t) we let $M_1^n = M^n \setminus \mathrm{Int}\, D^n$ and consider the diagram



where all the lines are Kan fibrations, (1), (2) by the *ambient isotopy theorem* and (3) and (4) by elementary homotopy arguments. It is well known that $\mathrm{E}^t(D^n, M^n) \to \mathrm{R}^t(D^n, M^n)$ and $\mathrm{E}^d(D^n, M^n) \to \mathrm{R}^d(D^n, M^n)$ are homotopy equivalences.

Because $\pi_i(O_n) = \pi_i(\mathrm{Top}_n)$ for $i = 0, 1$ and $n \geq 1$ we derive $\pi_1(\mathrm{E}^t(D^n, M^n), \theta\mathrm{E}^d(D^n, M^n), x) = 0$ for any $x \in \mathrm{E}^d(D^n, M^n)$. Therefore, using ($1^t$) we obtain ($2^t$), the proof of ($2^{pl}$) is exactly the same.

In order to prove (3^t) we denote by M_1^{n-1} the differentiable manifold $\partial M^n \setminus \mathrm{Int}\, N^{n-1}$ and consider the following commutative diagram

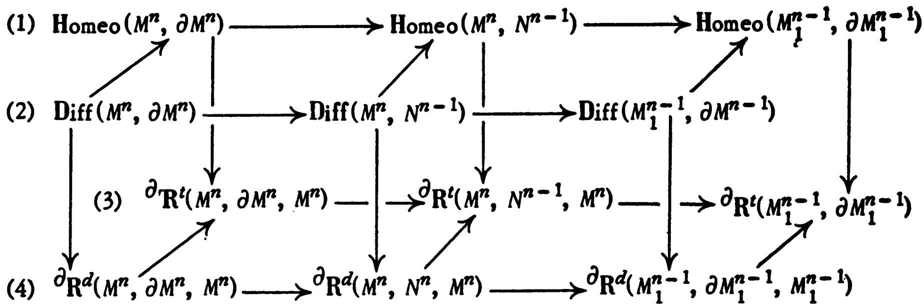


Figure 7

All the lines are principal fibrations for obvious reasons (all the semisimplicial complexes are groups) and all the maps group-homeomorphisms, etc. This diagram induces the obvious commutative diagram

$$\begin{array}{lccccc}
(1) & \mathrm{Homeo}(\cdots)/\mathrm{Diff}(\cdots) & \longrightarrow & \mathrm{Homeo}(\cdots)/\mathrm{Diff}(\cdots) & \longrightarrow & \mathrm{Homeo}(\cdots)/\mathrm{Diff}(\cdots) \\
 & \Big\downarrow \delta_1^{\cdots} & & \Big\downarrow \delta_2^{\cdots} & & \Big\downarrow \delta_3^{\cdots} \\
(2) & {}^{\partial}\mathrm{R}^{t}(\ldots)/{}^{\partial}\mathrm{R}^{d}(\ldots) & \longrightarrow & {}^{\partial}\mathrm{R}^{t}(\ldots)/{}^{\partial}\mathrm{R}^{d}(\ldots) & \longrightarrow & {}^{\partial}\mathrm{R}^{t}(\ldots)/{}^{\partial}\mathrm{R}^{d}(\ldots)
\end{array}$$

Figure 8

with (1) and (2) Kan fibrations. $\delta_1^{\cdots}$ and $\delta_3^{\cdots}$ induce an isomorphism for all homotopy groups π_i, $i \geq 1$, by (1^t) if $\partial N \neq \emptyset$, or by (1^t) and (2^t) if $\partial N = \emptyset$. We then obtain $\delta_2^{\cdots}$ induces an isomorphism for all homotopy groups in dimension $i \geq 2$.

The isomorphism for π_1 follows as soon as we prove $\delta_1^{\cdots}$ induces an injective correspondence on connected components, but this is the case according to standard smoothing theory as in [17].

The proof of (3^{pl}) goes on the same lines, namely we consider the corresponding diagram with ${}^{\partial}\mathrm{R}^{pd}(\cdots)$ instead of ${}^{\partial}\mathrm{R}^{t}(\cdots)$. The line (3) in Figure 7 will be replaced by

$$^{\partial}\mathrm{R}^{pd}(\ldots) \to {}^{\partial}\mathrm{R}^{pd}(\ldots) \to {}^{\partial}\mathrm{R}^{pd}(\ldots),$$

which although it is not a principal fibration is a Kan fibration. The corresponding diagram to Figure 8 still makes sense because $\mathrm{R}^{d}(\cdots)$ operates freely on $\mathrm{R}^{pd}(\cdots)$ as indicated before. Q.E.D.

Remark. For a pl manifold $|M^n|$ we can get a similar statement in comparing $\mathrm{Homeo}(\cdots)/\mathrm{PL}(\cdots)$ with $\Gamma(P^{t,pl}, s)$ in which case $P^{t,pl}$ denotes the $\{\mathrm{Top}_n/\{\mathrm{PL}_n\}$ bundle associated to the princiapl topological tangent bundle of $|M^n|$. This proof goes exactly in the same way as in the case top-diff. We did not pay special attention to this case because we are not particularly intérested in it.

Theorem 4.4. (a) $\mathrm{Diff}(D^n, \partial D^n)$ *is homotopy equivalent to* $\Omega^{n+1}(\{\mathrm{PL}_n/O_n\})$ *as topological groups.*

(b) *If* $n \neq 4$, $\mathrm{Diff}(D^n, \partial D^n)$ *is homotopy equivalent to* $\Omega^{n+1}(\{\mathrm{Top}_n/O_n\})$ *as topological groups.*

(c) $\mathrm{Diff}(D^n, D_+^{n-1})$ *is homotopy equivalent to* $\Omega^n(E(\{\mathrm{PL}_n/O_n\}, \{\mathrm{PL}_{n-1}/O_{n-1}\}, x))$.

(d) *If* $n \neq 4, 5$, $\mathrm{Diff}(D^n, D_+^{n-1})$ *is homotopy equivalent to* $\Omega^n(E(\{\mathrm{Top}_n/O_n\}, \{\mathrm{Top}_{n-1}/O_{n-1}\}, x))$.

Recall that for a pair of topological spaces (B, A), $A \subset B$, one defines $E(B, A, x)$, $x \in A$, as the space of all continuous paths $f: [0, 1] \to B$ with $f(0) = *$ and $f(1) \in A$.

Proof of Theorem 4.4. It is clear that (because D^n is parallelisable) $\Gamma(P^t)$ identifies to $\mathrm{Maps}(D^n, \{\mathrm{PL}_n/O_n\})$ with compact open topology. The section s via this identification is the constant map. Therefore $\Gamma^{S^n}(P^t)$ identifies to $\mathrm{Maps}(D^n, \partial D^n, \{\mathrm{PL}_n/O_n\}, x) = \Omega^n(\{\mathrm{PL}_n/O_n\})$. From Alexander's trick which claims $\mathrm{PD}(D^n, \partial D^n)$ is contractible, applying Theorem 4.2(1^{pl}), one obtains an IHE-map from $B\,\mathrm{Diff}(D^n, \partial D^n) \to \Omega^n(\{\mathrm{PL}_n/O_n\}, x)$ which sends the canonical basepoint of $B\,\mathrm{Diff}(D^n, \partial D^n)$ into the canonical basepoint of $\Omega^n(\cdots)$. Then (a) follows. Exactly the same argument establishes (b).

We will prove now (d). Clearly $\Gamma^{D^{n-1}_+}(P^t, {}^{\partial}P^t, s)$ is a subspace of $\Gamma(P^t, {}^{\partial}P^t_{\dot{D}^{n-1}})$ which identifies to the space of continuous maps $(D^n, D^{n-1}) \to (\{\mathrm{Top}_n/O_n\}, \{\mathrm{Top}_{n-1}/O_n\})$. By this identification, the section s represents the constant map. Then $\Gamma^{D^{n-1}_+}(P^t, {}^{\partial}P^t, s)$ identifies to the space of continuous maps $f: (D^n, D^{n-1}) \to (\{\mathrm{Top}_n/O_n\}, \{\mathrm{Top}_{n-1}/O_{n-1}\})$ with $f\,|D^n_+ = s$. It is an easy exercise to show that this space identifies to $\Omega^{n-1}E(\{\mathrm{Top}_n/O_n\}, \{\mathrm{Top}_{n-1}/O_{n-1}\})$. The same Alexander trick allows us to show $\mathrm{Homeo}(D^n, D^n_+)$ is contractible. Following the same arguments as in the proof of (a) and (b) one gets (d). Exactly the same arguments can be repeated in order to obtain (c). Q.E.D.

Appendix to §4. In the introduction to this paper we have considered the simplicial groups (Δ-groups with the terminology of [22]), $\widetilde{\mathrm{Diff}}(M, \partial M)$, $\widetilde{\mathrm{Homeo}}(M^n, \partial M^n)$ and for a differentiable triangulation $t: |M^n| \to M^n$, the simplicial complex (Δ-set) $\widetilde{\mathrm{PD}}(M^n, t, \partial M^n)$. $\widetilde{\mathrm{Diff}}(M, \partial M)$ acts freely on $\widetilde{\mathrm{PD}}(M^n, t, \partial M^n)$ by left composition. We can also consider the simplicial (Δ-sets) semigroups $\tilde{R}^{\cdots}(M^n, \partial M^n, M^n)$ whose k-simplexes are germs of bundle representations $\sigma: \Delta[k] \times r^s(M) \to \Delta[k] \times r^s(M)$ which restricts to identity on $\Delta[k] \times r \mid \partial M$ (to $\widetilde{\delta t}$ in pd-case) and are "face preserving" i.e., $\sigma(d_i \Delta[k] \times r^s(M)) \subset d_i \Delta[k] \times r^s(M)$. Here $r^s(M)$ denotes the stable tangent vector bundle, topological microbundle or pl microbundle; the condition to be "face preserving" replaces the condition to commute with the projection on $\Delta[k]$ in the definition of $R^{\cdots}(\cdots)$. $\tilde{R}^d(\cdots)$ and $\tilde{R}^t(\cdots)$ are simplicial semigroups and $\tilde{R}^d(\cdots)$ acts freely on $\tilde{R}^{pd}(\cdots)$. There are natural inclusions

(1) $S\,\mathrm{Homeo}(M^n, \partial M^n) \hookrightarrow \widetilde{\mathrm{Homeo}}(M^n, \partial M^n)$,

(2) $S^d\mathrm{Diff}(M^n, \partial M^n) \subset \widetilde{\mathrm{Diff}}(M^n, \partial M^n)$,

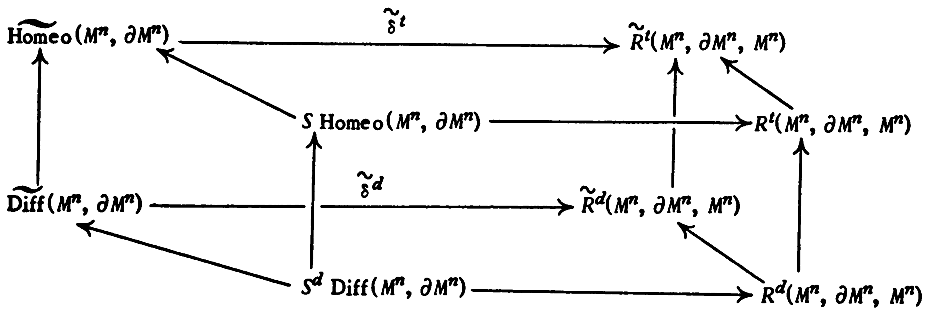
(3) $\mathrm{PD}(M^n, t, \partial M^n) \subset \widetilde{\mathrm{PD}}(M^n, t, \partial M^n)$,

(4) $R^d(M^n, \partial M^n, M^n) \subset \tilde{R}^d(M^n, \partial M^n, M^n)$,

(5) $R^t(M^n, \partial M^n, M^n) \subset \tilde{R}^t(M^n, \partial M^n, M^n)$,

(6) $R^{pd}(M^n, t, \partial M^n, M^n) \subset \tilde{R}^{pd}(M^n, t, \partial M^n, M^n)$.

(1), (2), (4), (5) are homomorphisms; (3) and (6) are compatible with the action of $S^d\mathrm{Diff}(M^n, \partial M^n)$ and $\widetilde{\mathrm{Diff}}(M^n, \partial M^n)$ on $\mathrm{PD}(M^n, t, \partial M^n)$ and $\widetilde{\mathrm{PD}}(M^n, t, \partial M^n)$ respectively, of $R^d(M^n, \partial M^n, M^n)$ and $\tilde{R}^d(M^n, \partial M^n, M^n)$ on $R^{pd}(M^n, t, \partial M^n, M^n)$ and $\tilde{R}^{pd}(M^n, t, \partial M^n, M^n)$ respectively. The "derivative" maps $\tilde{\delta}^d, \tilde{\delta}^t, \tilde{\delta}^{pd}$ are also defined, $\tilde{\delta}^d$ and $\tilde{\delta}^t$ being homomorphisms, $\tilde{\delta}^{pd}$ being compatible with δ^d with respect to the actions of $\widetilde{\mathrm{Diff}}(\ )$ and $\tilde{R}^d(\ )$ on $\mathrm{PD}(\ )$ and $\tilde{R}^{pd}(\ )$. It is clear that the diagram



and the corresponding PD-Diff diagram is commutative. We also have considered the spaces of sections $\Gamma^{\partial M}(\bar{P}^t, s)$ and $\Gamma^{\partial M}(\bar{P}^{pl}, s)$ of the Top/O and PL/O bundles associated to the principal Top_n and PL_n bundle of M^n and $|M^n|$. Because of the theory of principal fibrations in the category of Δ-sets [21] still holds, we can repeat "word by word" the arguments at the end of §1 and using the identification of $\tilde{R}^t(\cdots)/\tilde{R}^d(\cdots)$ to some part of $S\Gamma^{\partial M}(\bar{P}^t, s)$ respectively of $S\Gamma^{\partial M}(\bar{P}^{pl}, s)$ to some part of $S\Gamma^{\partial M}(\bar{P}^{pl}, s)$, and we get the following homotopy commutative diagrams:

$$\begin{array}{ccc}
\widetilde{\mathrm{Homeo}}(M, \partial M)/\widetilde{\mathrm{Diff}}(M, \partial M) & \xrightarrow{\tilde{\delta}^{t,d}} & S\Gamma^{\partial M}(\bar{P}^t, s) \\
\uparrow & & \uparrow \\
S\,\mathrm{Homeo}(M, \partial M)/S^d\,\mathrm{Diff}(M, \partial M) & \xrightarrow{\delta^{t,d}} & S\Gamma^{\partial M}(P^t, s)
\end{array}$$

$$\begin{array}{ccc}
\widetilde{\mathrm{PD}}(M, \partial M)/\widetilde{\mathrm{Diff}}(M, \partial M) & \xrightarrow{\tilde{\delta}^{pl,d}} & S\Gamma^{\partial M}(\bar{P}^{pl}, s) \\
\uparrow & & \uparrow \\
\mathrm{PD}(M, \partial M)/S^d\,\mathrm{Diff}(M, \partial M) & \xrightarrow{\delta^{pl,d}} & S\Gamma^{\partial M}(P^{pl}, s)
\end{array}$$

It has been shown in [2] (in slightly different words) that $\{\delta^{pl,d}\}$ and $\{\delta^{t,d}\}$, the geometric realizations of $\tilde{\delta}^{pl,d}$ and $\tilde{\delta}^{t,d}$, are homotopy equivalences (the second only if $n \neq 4$ or 5 or $n = 5$ and $\partial M^5 = S^4$). The Alexander trick, applied again, shows $\{\widetilde{\text{Homeo}}(D^n, \partial D^n)\}$ and $\{\widetilde{\text{PD}}(D^n, t, \partial D^n)\}$ are contractible. Therefore we can complete Theorem 4.4(a) with the following

Theorem 4.5 (Ap.). *There exist the following homotopy commutative diagrams*

$$\begin{array}{ccc} \{S^d\ \text{Diff}(D^n, \partial D^n)\} & \longrightarrow & \Omega^{n+1}\{(\text{PL}_n/O_n)\} \\ \Big\downarrow i & \text{I} & \Big\downarrow \Omega^{n+1}j^n \\ \{\widetilde{\text{Diff}}(D^n, \partial D^n)\} & \longrightarrow & \Omega^{n+1}\{(\text{PL}/O)\} \end{array}$$

$$\begin{array}{ccc} S^d\ \text{Diff}(D^n, \partial D^n) & \longrightarrow & \Omega^{n+1}(\text{Top}_n/O_n) \\ \Big\downarrow i & \text{II} & \Big\downarrow \Omega^{n+1}j^n \\ \widetilde{\text{Diff}}(D^n, \partial D^n) & \longrightarrow & \Omega^{n+1}(\text{Top}/O) \end{array}$$

such that the horizontal arrows are homotopy equivalences for any n in diagram I *and for any $n \neq 5$ in diagram* II. *(j^n denotes the canonical inclusions* $\text{Top}_n/O_n \to \text{Top}/O$ *or* $\text{PL}_n/O_n \to \text{PL}/O$.)

Then Cerf's results [5] imply $\pi_0(\text{Diff}(D^n, \partial D^n)) \to \pi_0(\widetilde{\text{Diff}}(D^n, \partial D^n))$ is bijective and $\pi_1(\text{Diff}(D^n, \partial D^n)) \to \pi_1(\widetilde{\text{Diff}}(D^n, \partial D^n))$ is surjective as soon as $n \geq 5$, hence we have

Theorem 4.6 (Ap.). *If $n \geq 5$, the natural inclusions*

$$\text{PL}_n/O_n \xrightarrow{j^n} \text{PL}/O \quad \text{and} \quad \text{Top}_n/O_n \xrightarrow{j^n} \text{Top}/O$$

induce for homotopy an isomorphism in dimension $n+1$ and an epimorphism in dimension $n+2$.

REFERENCES

1. T. Akiba, PL_3 *is homotopy equivalent to* O_3, Mimeographed, SUNY.

2. P. L. Antonelli, D. Burghelea and P. J. Kahn, *The non finite-homotopy type of some diffeomorphism groups*, Topology 11 (1972), 1–49. MR 45 #1193.

3. ———, *The concordance-homotopy groups of geometric automorphism groups*, Lecture Notes in Math., vol. 215, Springer-Verlag, Berlin and New York.

4. D. Burghelea and N. Kuiper, *Hilbert manifolds*, Ann. of Math. (2) 90 (1969), 379–417. MR 40 #6589.

5. J. Cerf, *La stratification naturelle des espaces des functions différentielles réelles et le théorème de pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. No. 39, (1970).

6. J. Cerf, *Invariants des paires d'espaces. Applications à la topologie différentielle*, Topologia Differenziale (Centro Internaz. Mat. Estivo, 1° Ciclo, Urbino, 1962), Lezione 1, Edizioni Cremonese, Rome, 1963, 24 pp. MR 29 #2813.

7. C. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry 3 (1969), 19–43. MR 43 #2737a.

8. R. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Ann. of Math. (2) 93 (1971), 63–88. MR 44 #1032.

9. A Haefliger and V. Poenaru, *La classification des immersions combinatoires*, Inst. Hautes Études Sci. Publ. Math. No. 23 (1964), 75–91. MR 30 #2515.

10. A. Haefliger, *Proceedings of Liverpool Singularities Symposium* II, Lecture Notes in Math., vol. 209, Springer-Verlag, Berlin and New York, 1971.

11. M. E. Hamstron, *Homotopy groups of the space of homeomorphisms in a 2-manifold*, Illinois J. Math. 10 (1966), 563–573. MR 34 #2014.

12. A. Hatcher, *Parameterized h-cobordism theory*, Colloque International d'Analyse et Topologie Differentielles, Strasbourg, June 1972.

13. D. Henderson, *Infinite-dimensional manifolds are open sets of Hilbert space*, Topology 9 (1969), 25–33. MR 40 #3581.

14. J. Hudson, *Piecewise linear topology*, University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees, Benjamin, New York, 1969. MR 40 #2094.

15. R. Kirby and L. Siebenmann, *On the triangulation of manifolds and Hauptvermutung*, Bull. Amer. Math. Soc. 75 (1969), 742–749. MR 39 #3500.

16. H. Kneser, *Die Deformationssatze der einfach zusammenhangen Flacher*, Math. Z. 25 (1926), 362–372.

17. R. Lashof and M. Rothenberg, *Microbundles and smoothing*, Topology 3 (1965), 357–388. MR 31 #752.

18. R. Lashof, *The immersion approach to triangulation and smoothing*, Proc. Sympos. Pure Math., vol. 22, Amer. Math. Soc., Providence, R. I., 1971, pp. 131–164.

19. J. A. Lees, *Immersions and surgeries of topological manifolds*, Bull. Amer. Math. Soc. 75 (1969), 529–534. MR 39 #959.

20. C. Morlet, *Isotopie et pseudo-isotopie*, C. R. Acad. Sci. Paris Sér. A-B 266 (1968), A559–A560. MR 38 #5228.

21. J. Munkres, *Elementary differential topology*, rev. ed., Ann. of Math. Studies, no. 54, Princeton Univ. Press, Princeton, N. J., 1966. MR 33 #6637.

22. C. Rourke and B. Sanderson, Δ *sets*, (mimeo) (to appear).

23. S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. 10 (1959), 621–626. MR 22 #3004.

24. H. Toda, *p-primary components of homotopy groups*. IV, Mem. Coll. Sci. Univ. Kyoto, vol. 37, pp. 297–337.

25. J. Wagoner, *Algebraic invariants for pseudo-isotopies*, Proceedings of Liverpool Singularities Symposium II, Lecture Notes in Math., vol. 209, Springer-Verlag, Berlin and New York, 1971.

26. J. H. C. Whitehead, *On C^1-complexes*, Ann. of Math. (2) 41 (1940), 809–824. MR 2, 73.

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